

# Existence of conformal metrics with constant scalar curvature and constant boundary mean curvature on compact manifolds

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## Abstract

We study the problem of deforming a Riemannian metric to a conformal one with nonzero constant scalar curvature and nonzero constant boundary mean curvature on a compact manifold of dimension  $n \geq 3$ . We prove the existence of such conformal metrics in the cases of  $n = 6, 7$  or the manifold is spin and some other remaining ones left by Escobar. Furthermore, in the positive Yamabe constant case, by normalizing scalar curvature to be 1, there exists a sequence of conformal metrics such that their constant boundary mean curvatures go to  $+\infty$ .

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## 1 Introduction

Analogous to the Yamabe problem, a very natural question on a compact manifold with boundary is for dimension  $n \geq 3$ , whether it is possible to deform any Riemannian metric to a conformal metric with constant scalar curvature and constant mean curvature curvature on the boundary. When studying the above problem, we benefited much from the series of great works on the Yamabe problem by Yamabe, Trudinger, Aubin and Schoen. We refer to [26, 7] for more background on the Yamabe problem.

Let  $(M, g_0)$  be a smooth compact Riemannian manifold of dimension  $n \geq 3$  with boundary  $\partial M$ . The problem is equivalent to finding a positive solution to the following PDE:

$$\begin{cases} -\frac{4(n-1)}{n-2}\Delta_{g_0}u + R_{g_0}u = c_1u^{\frac{n+2}{n-2}}, & \text{in } M, \\ \frac{2}{n-2}\frac{\partial u}{\partial \nu_{g_0}} + h_{g_0}u = c_2u^{\frac{n}{n-2}}, & \text{on } \partial M, \end{cases} \quad (1.1)$$

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where  $c_1, c_2 \in \mathbb{R}$ ,  $R_{g_0}$  is the scalar curvature,  $h_{g_0}$  is the mean curvature and  $\nu_{g_0}$  is the outward unit normal on  $\partial M$ . When  $c_1 = 0$  and  $c_2 \in \mathbb{R}$ , we refer the scalar-flat metrics of constant mean curvature problem to [19, 27, 28, 9, 13, 2, 1, 14]. When  $c_1 \in \mathbb{R}$ ,  $c_2 = 0$ , we refer the Yamabe problem with boundary to [21, 12, 9, 14, 4]. When  $c_1, c_2 \neq 0$ , problem (1.1) is also called constant scalar curvature and constant mean curvature problem. Escobar initiated the investigation of this problem in [22, 20]. In the subsequent papers [23, 24], Z. C. Han and Y. Y. Li proposed the following (weak version) conjecture:

**Conjecture (Han-Li).** *If  $Y(M, \partial M) > 0$ , problem (1.1) is solvable for any positive constant  $c_1$  and any  $c_2 \in \mathbb{R}$ .*

They proved that the conjecture is true when one of the following assumptions is fulfilled:

- (a)  $n \geq 5$  and  $\partial M$  admits at least one non-umbilic point (cf. [23]);
- (b)  $n \geq 3$  and  $(M, g_0)$  is locally conformally flat with umbilic boundary  $\partial M$  (cf. [24]).

Before presenting our results, we need to introduce natural conformal invariants. The (generalized) Yamabe constant  $Y(M, \partial M)$  is defined as

$$Y(M, \partial M) := \inf_{g \in [g_0]} \frac{\int_M R_g d\mu_g + 2(n-1) \int_{\partial M} h_g d\sigma_g}{\left(\int_M d\mu_g\right)^{\frac{n-2}{n}}}. \quad (1.2)$$

Similarly, we define (cf. [19])

$$Q(M, \partial M) := \inf_{g \in [g_0]} \frac{\int_M R_g d\mu_g + 2(n-1) \int_{\partial M} h_g d\sigma_g}{\left(\int_M d\sigma_g\right)^{\frac{n-2}{n-1}}}.$$

It was first pointed out by Zhiren Jin (cf. [18]) that  $Q(M, \partial M)$  could be  $-\infty$ , meanwhile  $Y(M, \partial M) > -\infty$ . If  $Y(M, \partial M) > (=)0$ , then there exists a conformal metric of  $g_0$  with zero scalar curvature in  $M$  and positive (zero) mean curvature on  $\partial M$ .<sup>1</sup> Furthermore,  $Y(M, \partial M) > 0$  if and only if  $Q(M, \partial M) > 0$ .

We remark that problem (1.1) is variational. The total scalar curvature plus total mean curvature functional is given by

$$E[u] = \int_M \left( \frac{4(n-1)}{n-2} |\nabla u|_{g_0}^2 + R_{g_0} u^2 \right) d\mu_{g_0} + 2(n-1) \int_{\partial M} h_{g_0} u^2 d\sigma_{g_0}. \quad (1.3)$$

Given any  $a, b > 0$ , we define a conformal invariant on compact manifolds with boundary by

$$Y_{a,b}(M, \partial M) = \inf_{g \in [g_0]} \frac{\int_M R_g d\mu_g + 2(n-1) \int_{\partial M} h_g d\sigma_g}{a \left(\int_M d\mu_g\right)^{\frac{n-2}{n}} + 2(n-1)b \left(\int_{\partial M} d\sigma_g\right)^{\frac{n-2}{n-1}}}$$

<sup>1</sup>From [21, Lemma 1.1], there exists  $g_1 \in [g_0]$  such that  $R_{g_1} > (=)0$  and  $h_{g_1} = 0$ . Let  $\varphi$  be a positive smooth minimizer of  $\{\int_M (\frac{4(n-1)}{n-2} |\nabla \psi|_{g_1}^2 + R_{g_1} \psi^2) d\mu_{g_1}; \psi \in H^1(M, g_1), \int_{\partial M} \psi^2 d\sigma_{g_1} = 1\}$ , then  $\varphi^{4/(n-2)} g_1$  is the desired conformal metric.

$$= \inf_{0 \neq u \in H^1(M, g_0)} \mathcal{Q}_{a,b}[u],$$

where

$$\mathcal{Q}_{a,b}[u] = \frac{E[u]}{a \left( \int_M |u|^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{\frac{n-2}{n}} + 2(n-1)b \left( \int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{\frac{n-2}{n-1}}}.$$

For  $n \geq 3$ , let  $\mathbb{R}_+^n = \{y = (y^1, \dots, y^n) \in \mathbb{R}^n; y^n > 0\}$  denote the half space. The next theorem gives a criterion for the existence of a minimizer for  $Y_{a,b}(M, \partial M)$ , which is attained by subcritical approximations.

**Theorem 1.1.** *Suppose  $Y_{a,b}(M, \partial M) < Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})$  for some given  $a, b > 0$ , then  $Y_{a,b}(M, \partial M)$  can be achieved by a positive smooth minimizer.*

In [5, 6] Araujo also gave some characterization of critical points (including minimizers) of  $E[u]$  under Escobar's non-homogeneous constraint (cf. [20]).

In order to apply Theorem 1.1 in the case of  $Y(M, \partial M) > 0$ , we need to construct a global test function  $\bar{U}_{(x_0, \epsilon)}$  as a small perturbation of a bubble function  $W_\epsilon$  with  $x_0 \in \partial M$  and small  $\epsilon > 0$ , such that  $\mathcal{Q}_{a,b}[\bar{U}_{(x_0, \epsilon)}] < Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})$ . We would like to mention some developments on the technique of constructing test functions in very closely related works. In dimension  $n \geq 6$ , Brendle [11] initiated this technique of constructing test functions through his study of the Yamabe flow. Subsequently Brendle and S. Chen [12] developed it to study the Yamabe problem with umbilic minimal boundary (i.e.  $c_1 \in \mathbb{R}, c_2 = 0$ ). Not long after that S. Chen [13] adapted the same technique to scalar-flat and constant mean curvature problem with umbilic boundary (i.e.  $c_1 = 0, c_2 \in \mathbb{R}$ ). One of the key ingredients in Almaraz [2] and Almaraz-L. Sun [4] is to extend such a technique to both the boundary  $\partial M$  has one non-umbilic point and the case of lower dimensions  $3 \leq n \leq 5$ . The correction term  $\psi$  in our test function (cf. (5.29)) origins from the linearization of scalar curvature and mean curvature at a round metric on a spherical cap, which has constant sectional curvature 4 (cf. Proposition 5.1).

We will use a notion of a *mass* associated to manifolds with boundary.

**Definition 1.2.** Let  $(N, g)$  be a Riemannian manifold with a boundary  $\partial N$ . We say that  $N$  is *asymptotically flat* with order  $p > 0$ , if there exist a compact set  $N_0 \subset N$  and a diffeomorphism  $F : N \setminus N_0 \rightarrow \mathbb{R}_+^n \setminus B_1^+(0)$  such that, in the coordinate chart defined by  $F$  (called *asymptotic coordinates* of  $N$ ), there holds

$$|g_{ij}(y) - \delta_{ij}| + |y| |\partial_k g_{ij}(y)| + |y|^2 |\partial_{kl}^2 g_{ij}(y)| = O(|y|^{-p}), \text{ as } |y| \rightarrow \infty,$$

where  $i, j, k, l = 1, \dots, n, B_1^+(0) = B_1(0) \cap \mathbb{R}_+^n$ .

Provided that the following limit

$$m(g) := \lim_{R \rightarrow \infty} \left[ \int_{\{y \in \mathbb{R}_+^n; |y|=R\}} \sum_{i,j=1}^n (g_{ij,j} - g_{jj,i}) \frac{y^i}{|y|} d\sigma + \int_{\{y \in \mathbb{R}^{n-1}; |y|=R\}} \sum_{a=1}^{n-1} g_{na} \frac{y^a}{|y|} d\sigma \right]$$

exists, we call it the *mass* of  $(N, g)$ . Moreover,  $m(g)$  is a geometric invariant in the sense that it is independent of asymptotic coordinates. The definition of the mass  $m(g)$  was first proposed by Marques. The following positive mass type conjecture was given in [2, 3].

**Conjecture (Positive mass with a non-compact boundary).** *If  $R_g, h_g \geq 0$ , then we have  $m(g) \geq 0$  and the equality holds if and only if  $N$  is isometric to  $\mathbb{R}_+^n$ .*

For  $n \geq 3$ , let  $d = [(n - 2)/2]$ . As in [2], we define

$$\mathcal{Z} = \{x_0 \in \partial M; \limsup_{x \rightarrow x_0} d_{g_0}(x, x_0)^{2-d} |W_{g_0}(x)|_{g_0} = 0 \text{ and} \\ \limsup_{x \rightarrow x_0} d_{g_0}(x, x_0)^{1-d} |\mathring{\pi}_{g_0}(x)|_{g_0} = 0\},$$

where  $W_{g_0}$  denotes the Weyl tensor in  $M$ ,  $\pi_{g_0}$  the second fundamental form and  $\mathring{\pi}_{g_0}$  the trace-free second fundamental form on  $\partial M$ . Then  $\mathcal{Z}$  only depends on the conformal structure of  $g_0$ , since  $W_{g_0}$  and  $\mathring{\pi}_{g_0}$  are both pointwise conformal invariants. In particular,  $\mathcal{Z} = \partial M$  when  $n = 3$ . Moreover, if the scalar curvature and the mean curvature are integrable on  $M$  and  $\partial M$  respectively and the decay order is  $p > (n - 2)/2$ , the *mass*  $m(g_0)$  is well-defined. This is the case for  $g_{x_0}$  when  $x_0 \in \mathcal{Z}$ .

For  $x_0 \in \partial M$ , let  $g_{x_0} \in [g_0]$  be the metric induced by the conformal Fermi coordinates around  $x_0$  (cf. [28]). Denote by  $G_{x_0}$  the Green's function of conformal Laplacian of  $g_{x_0}$  with pole at  $x_0$ , satisfying the boundary condition  $\partial_{\nu_{g_{x_0}}} G_{x_0} - \frac{n-2}{2} h_{g_{x_0}} G_{x_0} = 0$  on  $\partial M \setminus \{x_0\}$  (cf. (5.30)). Let  $\bar{g}_{x_0} = G_{x_0}^{4/(n-2)} g_{x_0}$ . Now we are ready to state our main result.

**Theorem 1.3.** *Let  $(M, g_0)$  be a smooth compact Riemannian manifold of dimension  $n \geq 3$  with boundary. Suppose that  $M$  is not conformally equivalent to  $\mathbb{R}_+^n$ . If  $Y(M, \partial M) > 0$ , assume either  $\partial M \setminus \mathcal{Z} \neq \emptyset$  or  $m(\bar{g}_{x_0}) > 0$  for some  $x_0 \in \mathcal{Z}$ , then*

$$Y_{a,b}(M, \partial M) < Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1}).$$

We should point out that such assumptions on compact manifolds in Theorem 1.3 (or with some minor modifications) have been used in some closely related problems, for instance, Brendle [11] for the Yamabe flow in dimension  $n \geq 6$ , S. Chen [13] and Almaraz [2] for  $c_1 = 0, c_2 \in \mathbb{R}$ , Brendle-Chen [12] and Almaraz-L. Sun [4] for  $c_1 \in \mathbb{R}, c_2 = 0$ .

Recent advances in the above positive mass type theorem have played an important role in such conformal curvature problems (cf. [3, 2, 29] etc.). As a direct consequence of Theorem 1.3 and the positive mass type theorem proved in [3], we obtain

**Theorem 1.4.** *Let  $(M, g_0)$  be a smooth compact Riemannian manifold of dimension  $n \geq 3$  with boundary. Suppose that  $M$  is not conformally equivalent to  $\mathbb{R}_+^n$  and  $Y(M, \partial M) > 0$ . Assume that one of the following assumptions is satisfied:*

- (i)  $\partial M \setminus \mathcal{Z} \neq \emptyset$ ;
- (ii)  $3 \leq n \leq 7$  or  $M$  is spin;
- (iii)  $n \geq 8$  and  $(M, g_0)$  is locally conformally flat with umbilic boundary  $\partial M$ .

Then given any  $a, b > 0$ , there exists at least one positive smooth minimizer  $u_{a,b}$  for  $Y_{a,b}(M, \partial M)$ . Moreover, the conformal metric  $u_{a,b}^{4/(n-2)} g_0$ , modulo a positive constant multiple, has scalar curvature 1 and some positive constant boundary mean curvature.

When  $Y(M, \partial M) > 0$ , Escobar proved in [20, Theorem 4.2] the existence of such conformal metrics in Theorem 1.4 under one of the following hypotheses:

- (1)  $3 \leq n \leq 5$ ;
- (2)  $\partial M$  has at least a nonumbilic point;
- (3)  $\partial M$  is umbilic and either  $M$  is locally conformally flat or the Weyl tensor does not identically vanish on  $\partial M$ .

Then we generalize the existence results to the cases including  $n = 6, 7$  or  $M$  is spin. Some remaining cases left by Escobar are the manifolds that are not locally conformally flat and  $\partial M$  is umbilic, and Weyl tensor vanishes identically on  $\partial M$  and  $n \geq 6$ . Thus our Theorem 1.4 also generalizes to this type of manifolds under the assumption  $\partial M \setminus \mathcal{Z} \neq \emptyset$ . We next prove the compactness of the minimizers for  $Y_{a,b}(M, \partial M)$  when  $(a, b)$  varies in a compact set  $K$  of  $\{(a, b); a \geq 0, b \geq 0\} \setminus \{(0, 0)\}$ . We denote by  $\mathcal{M}_{a,b}$  the set of positive smooth minimizers of  $Y_{a,b}(M, \partial M)$  with the normalization (4.1).

**Theorem 1.5.** *Let  $K$  and  $\mathcal{M}_{a,b}$  as defined above. Suppose  $Y_{a,b}(M, \partial M) < Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})$  for all  $(a, b) \in K$ , then there exists  $C = C(K, g_0)$  such that*

$$C^{-1} \leq u_{a,b} \leq C, \quad \|u_{a,b}\|_{C^2(M)} \leq C, \quad \forall u_{a,b} \in \cup_{(a,b) \in K} \mathcal{M}_{a,b}.$$

It follows from Theorem 1.5 that in terms of normalized conformal metrics having scalar curvature 1, there exists a sequence of such conformal metrics such that their constant boundary mean curvatures go to  $+\infty$ . We refer the details to the end of Section 4. In contrast with our result, the constant mean curvature of such a conformal metric in [20, Theorem 4.2] only admits a small real number. Indeed, the smallness of  $b \in \mathbb{R}$  in a conformal invariant  $G_{a,b}(M)$  (see also Section 2) is very crucial in the proof of [20].

**Remark 1.6.** When  $Y(M, \partial M) < 0$ , as a direct consequence of [15, Theorem 1.1], there exists a conformal metric such that its scalar curvature equals  $-1$  and its boundary mean curvature equals any negative real number.

**Remark 1.7.** The assumptions (ii) and (iii) enable us to use the above positive mass type theorem results of [2] and the appendix of [21]. Due to similar technical reasons as the study of the Yamabe flow in [11], these existence results are reduced to the validity of the above positive mass type conjecture in higher dimensions  $n \geq 8$ .

In a forthcoming paper [16] we will adopt a geometric flow with another Escobar's non-homogeneous constraint used in [20] to tackle problem (1.1). In general, such a geometric flow can be used to find some non-minimizer critical points of the associated functional. More related conformal geometric flows can be referred to [10, 11, 9, 14, 15, 2] and the references therein.

The present paper is organized as follows. In Section 2, we describe some properties of the standard bubble on the boundary and conformal invariant  $Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})$ . In section 3, a procedure of subcritical approximations is set up to prove Theorem 1.1. The compactness of minimizers for  $Y_{a,b}(M, \partial M)$  with various  $a, b > 0$  is presented in Section 4. In section 5, we derive the detailed computations for the linearization of scalar curvature and mean curvature at a round metric on a spherical cap in Section 5.1, which is of independent interest. Finally in Section 5.2, we construct these desired test functions required by Theorem 1.1 and establish its energy estimate, then Theorem 1.3 follows.

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## 2 Preliminaries

Let  $T_c$  be a negative real number, it follows from the classification theorem in [25] that all nonnegative solutions to the following PDE

$$\begin{cases} -\Delta v = n(n-2)v^{\frac{n+2}{n-2}}, & \text{in } \mathbb{R}_+^n, \\ \frac{\partial v}{\partial y^n} = (n-2)T_c v^{\frac{n}{n-2}}, & \text{on } \mathbb{R}^{n-1}. \end{cases} \quad (2.1)$$

must be either  $v \equiv 0$  or  $v(y) = W(y)$  (up to dilations and translations in variables  $y^1, \dots, y^{n-1}$ ), where

$$W(y) = \left( \frac{1}{1 + |y - T_c \mathbf{e}_n|^2} \right)^{\frac{n-2}{2}}$$

and  $\mathbf{e}_n$  is the unit direction vector in  $n$ -th coordinate. In particular, we set

$$W_\epsilon(y) = \epsilon^{\frac{2-n}{2}} W(\epsilon^{-1}y) = \left( \frac{\epsilon}{\epsilon^2 + |y - T_c \epsilon \mathbf{e}_n|^2} \right)^{\frac{n-2}{2}}, \quad \forall \epsilon > 0, \quad (2.2)$$

which satisfy (2.1) and are also the extremal functions of the associated Sobolev inequality induced by  $Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})$  (cf. [22, Theorem 3.3] or Lemma 3.4 below).

For each fixed  $a, b > 0$ , any positive minimizers of  $Y_{a,b}(M, \partial M)$  satisfy

$$\begin{cases} -\frac{4(n-1)}{n-2} \Delta_{g_0} u + R_{g_0} u = \mu(M) a \left( \int_M u^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{-\frac{2}{n}} u^{\frac{n+2}{n-2}} & \text{in } M, \\ \frac{2}{n-2} \frac{\partial u}{\partial \nu_{g_0}} + h_{g_0} u = \mu(M) b \left( \int_{\partial M} u^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{-\frac{1}{n-1}} u^{\frac{n}{n-2}} & \text{on } \partial M, \end{cases} \quad (2.3)$$

where  $\mu(M) = Y_{a,b}(M, \partial M)$ .

When  $(M, g_0) = (\mathbb{R}_+^n, g_{\mathbb{R}^n})$ , problem (2.3) is equivalent to the solvability of positive solutions to

$$\begin{cases} -\frac{4(n-1)}{n-2}\Delta u = \mu a \left( \int_{\mathbb{R}_+^n} u^{\frac{2n}{n-2}} dx \right)^{-\frac{2}{n}} u^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}_+^n, \\ -\frac{2}{n-2} \frac{\partial u}{\partial y^n} = \mu b \left( \int_{\mathbb{R}^{n-1}} u^{\frac{2(n-1)}{n-2}} d\sigma \right)^{-\frac{1}{n-1}} u^{\frac{n}{n-2}} & \text{on } \mathbb{R}^{n-1}, \end{cases} \quad (2.4)$$

where  $\mu = \mu(\mathbb{R}_+^n) = Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})$ . A simple but vital observation is that if  $u$  is a smooth positive solution to problem (2.4), so is  $c_* u$  for all  $c_* \in \mathbb{R}_+$ . Hence all positive solutions to problem (2.4) are in the form of, up to dilations and translations in variables  $y^1, \dots, y^{n-1}$ ,

$$c_* \left( \frac{1}{1 + |y - T_c \mathbf{e}_n|^2} \right)^{\frac{n-2}{2}}$$

for all  $c_* > 0$  and some  $T_c < 0$  depending on  $n, a, b$ . We choose  $c_* = 1$  hereafter, namely, for this fixed  $T_c < 0$ , the associated function

$$W(y) = \left( \frac{1}{1 + |y - T_c \mathbf{e}_n|^2} \right)^{\frac{n-2}{2}}$$

is a positive solution to both problems (2.1) and (2.4).

Denote by a mapping  $\pi : S^n(T_c \mathbf{e}_n) \setminus \{T_c \mathbf{e}_n + \mathbf{e}_{n+1}\} \rightarrow \{\xi + T_c \mathbf{e}_n \in \mathbb{R}^{n+1}; \xi^{n+1} = 0\} \simeq \mathbb{R}^n$  the stereographic projection from the unit sphere  $S^n(T_c \mathbf{e}_n)$  in  $\mathbb{R}^{n+1}$  centered at  $T_c \mathbf{e}_n$ . Then for  $y \in \mathbb{R}_+^n$ , we set  $\xi = \pi^{-1}(y) \in S^n$ , namely (see also [23, (3.1) on page 831])

$$\begin{cases} \xi^a = \frac{2y^a}{1 + |y - T_c \mathbf{e}_n|^2}, & \text{for } 1 \leq a \leq n-1, \\ \xi^n = \frac{2(y^n - T_c)}{1 + |y - T_c \mathbf{e}_n|^2}, \\ \xi^{n+1} = \frac{|y - T_c \mathbf{e}_n|^2 - 1}{1 + |y - T_c \mathbf{e}_n|^2}. \end{cases}$$

Let  $\Sigma$  be a spherical cap (cf. Figure 1) equipped with a round metric  $\frac{1}{4}g_{S^n}$ , where  $g_{S^n}$  is the standard metric of the unit sphere  $S^n(T_c \mathbf{e}_n)$ . Then a direct computation shows

$$\frac{1}{4}(\pi^{-1})^*(g_{S^n}) = \left( \frac{1}{1 + |y - T_c \mathbf{e}_n|^2} \right)^2 g_{\mathbb{R}^n} = W(y)^{\frac{4}{n-2}} g_{\mathbb{R}^n}.$$

Denote by  $\omega_{n-1}$  the volume of the standard unit sphere in  $\mathbb{R}^n$ . Define

$$A = \int_{\mathbb{R}_+^n} W(y)^{\frac{2n}{n-2}} dy \quad \text{and} \quad B = \int_{\mathbb{R}^{n-1}} W(y)^{\frac{2(n-1)}{n-2}} d\sigma.$$

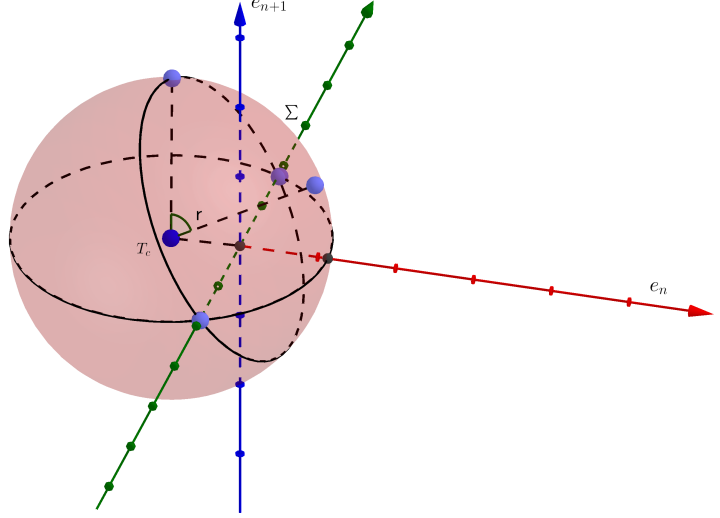


Figure 1: Stereographic projection for  $T_c < 0$

Notice that  $A, B$  only depend on  $n, T_c$ . Using (2.1) we get

$$\int_{\mathbb{R}_+^n} |\nabla W(y)|^2 dy = n(n-2)A - (n-2)T_c B. \quad (2.5)$$

Recall that, from [22, Theorem 3.3] that  $Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})$  can be achieved by  $W$  with some  $T_c$  (up to dilations and translations in variables  $y^1, \dots, y^{n-1}$ ) modulo a positive constant multiple. Comparing (2.4) and (2.1), as well as the above comments, we have

$$\mu \frac{n-2}{4(n-1)} a A^{-\frac{2}{n}} = n(n-2), \quad \mu \frac{n-2}{2} b B^{-\frac{1}{n-1}} = -(n-2)T_c,$$

whence

$$-a A^{-\frac{2}{n}} T_c = 2n(n-1) b B^{-\frac{1}{n-1}}.$$

Indeed we will establish that each pair of  $a, b > 0$  corresponds to a unique  $T_c$  satisfying the above identity.

**Lemma 2.1.** *Given any  $a, b > 0$ , there exists a unique  $T_c \in (-\infty, 0)$  such that*

$$-a A^{-\frac{2}{n}} T_c = 2n(n-1) b B^{-\frac{1}{n-1}}. \quad (2.6)$$

*In particular,  $T_c$  is a continuous function of  $(a, b) \in \mathbb{R}_+ \times \mathbb{R}_+$ . Moreover, for such a  $W$  satisfying (2.4) with the above unique  $T_c$ , there holds*

$$Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1}) = 4n(n-1) a^{-1} A^{\frac{2}{n}} = -2T_c b^{-1} B^{\frac{1}{n-1}}.$$



*Proof.* Let  $\cos r = \frac{-T_c}{\sqrt{1+T_c^2}}$ ,  $r \in (0, \frac{\pi}{2})$ , then  $A$  and  $B$  turn to

$$A(r) = \omega_{n-1} \int_0^r (\sin \tau)^{n-1} d\tau, \quad B(r) = \omega_{n-1} (\sin r)^{n-1}. \quad (2.7)$$

Then equation (2.6) is equivalent to finding some  $r \in (0, \frac{\pi}{2})$  such that

$$f(r) := 2n(n-1)bA^{\frac{2}{n}}B^{-\frac{1}{n-1}} - a \cot r = 0.$$

First it is easy to verify that

$$\lim_{r \searrow 0} f(r) = -\infty \quad \text{and} \quad \lim_{r \nearrow \frac{\pi}{2}} f(r) = \text{constant} > 0.$$

Next we claim that  $f(r)$  is increasing in  $(0, \frac{\pi}{2})$ . To see this, we have

$$\begin{aligned} \frac{d}{dr} \log(B^{-\frac{1}{n-1}} A^{\frac{2}{n}}) &= \frac{2}{n} \frac{A'}{A} - \frac{1}{n-1} \frac{B'}{B} \\ &= \frac{1}{\sin r \int_0^r (\sin \tau)^{n-1} d\tau} \left[ \frac{2}{n} (\sin r)^n - \cos r \int_0^r (\sin \tau)^{n-1} d\tau \right]. \end{aligned}$$

Observe that

$$\cos r \int_0^r (\sin \tau)^{n-1} d\tau \leq \int_0^r (\sin \tau)^{n-1} \cos \tau d\tau = \frac{1}{n} (\sin r)^n.$$

This implies  $(B^{-\frac{1}{n-1}} A^{\frac{2}{n}})(r)$  is increasing in  $(0, \frac{\pi}{2})$ , as well as is  $f(r)$ . Hence we conclude that there exists a unique  $r \in (0, \frac{\pi}{2})$  such that  $f(r) = 0$ , namely there exists a unique  $T_c < 0$  satisfying (2.6).

By [22, Theorem 3.3], (2.5) and (2.6), we get

$$\begin{aligned} Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1}) &= \frac{\frac{4(n-1)}{n-2} \int_{\mathbb{R}_+^n} |\nabla W|^2 dy}{aA^{\frac{n-2}{n}} + 2(n-1)bB^{\frac{n-2}{n-1}}} \\ &= \frac{4(n-1)}{n-2} \frac{n(n-2)A - (n-2)T_c B}{aA^{\frac{n-2}{n}} + 2(n-1)bB^{\frac{n-2}{n-1}}} \\ &= 4n(n-1)A^{\frac{2}{n}}a^{-1} = -2B^{\frac{1}{n-1}}T_cb^{-1}. \end{aligned} \quad (2.8)$$

In terms of the variable  $T_c$ , from (2.7) that  $A(T_c)$  is increasing in  $(-\infty, 0)$ . One may regard  $T_c$  as a function of  $(a, b)$ . Indeed one can show that  $Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})$  is continuous in  $(a, b) \in \mathbb{R}_+ \times \mathbb{R}_+$  (see e.g. Proposition 4.1 below). From this and the third identity in (2.8), we get  $A$  is a continuous function of  $(a, b)$ . Hence we conclude that  $T_c$  is a continuous function in  $(a, b)$ .  $\square$

From now on, we fix  $T_c < 0$  as the unique one in Lemma 2.1 without otherwise stated. In [20], Escobar introduced a conformal invariant by  $G_{a,b} = \inf \{E[u]; u \in C_{a,b}\}$ , where  $a > 0, b \in \mathbb{R}$  and

$$C_{a,b} = \left\{ u \in C^1(\bar{M}); a \int_M |u|^{\frac{2n}{n-2}} d\mu_{g_0} + b \int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} = 1 \right\}.$$

He established that  $G_{a,b}(M) \leq G_{a,b}(\mathbb{R}_+^n)$  holds for any compact Riemannian manifold with boundary. By similarly constructing a local test function as a perturbation of  $W_\epsilon$  under the Fermi coordinates around a boundary point, one can mimick the proof of [20, Proposition 3.1] to show  $Y_{a,b}(M, \partial M) \leq Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})$ . Since it is more or less standard to the experts in this field, we omit the details here.

### 3 Existence of minimizers

The purpose of this section is to establish Theorem 1.1. We adopt the method of subcritical approximations to realize it. For  $1 < q \leq \frac{n+2}{n-2}$ , we define

$$\mathcal{Q}_{a,b}^q[u] = \frac{E[u]}{a \left( \int_M |u|^{q+1} d\mu_{g_0} \right)^{\frac{2}{q+1}} + 2(n-1)b \left( \int_{\partial M} |u|^{\frac{q+3}{2}} d\sigma_{g_0} \right)^{\frac{4}{q+3}}}$$

for any  $u \in H^1(M, g_0)$ . Notice that  $\mathcal{Q}_{a,b}^q[u]$  always has a lower bound when  $Y(M, \partial M) \geq 0$ , we set

$$\mu_q = \inf_{0 \neq u \in H^1(M, g_0)} \mathcal{Q}_{a,b}^q[u].$$

For brevity, we use  $\mu_{(n+2)/(n-2)} = Y_{a,b}(M, \partial M)$  and  $\mathcal{Q}_{a,b}^{(n+2)/(n-2)}[u] = \mathcal{Q}_{a,b}[u]$ .

**Lemma 3.1.** *Given  $a, b > 0$ , there holds  $\limsup_{q \nearrow \frac{n+2}{n-2}} \mu_q \leq Y_{a,b}(M, \partial M)$ . Moreover, if  $Y(M, \partial M) \geq 0$ , there holds  $\lim_{q \nearrow \frac{n+2}{n-2}} \mu_q = Y_{a,b}(M, \partial M)$ .*

*Proof.* For any  $\epsilon > 0$ , there exists  $\bar{u} > 0$  such that  $\mathcal{Q}_{a,b}[\bar{u}] \leq Y_{a,b}(M, \partial M) + \epsilon$ . For each  $\bar{u}$ , there holds  $\lim_{q \nearrow \frac{n+2}{n-2}} \mathcal{Q}_{a,b}^q[\bar{u}] = \mathcal{Q}_{a,b}[\bar{u}]$ . Then we have

$$\limsup_{q \nearrow \frac{n+2}{n-2}} \mu_q \leq \limsup_{q \nearrow \frac{n+2}{n-2}} \mathcal{Q}_{a,b}^q[\bar{u}] \leq Y_{a,b}(M, \partial M) + \epsilon,$$

which gives the first assertion. If  $Y(M, \partial M) \geq 0$ , then  $E[u] \geq 0$  for any  $u \in H^1(M, g_0)$ . Notice that

$$\mathcal{Q}_{a,b}[u] = \mathcal{Q}_{a,b}^q[u] \frac{a \left( \int_M |u|^{\frac{q+2}{2}} d\mu_{g_0} \right)^{\frac{2}{q+1}} + 2(n-1)b \left( \int_{\partial M} |u|^{\frac{q+3}{2}} d\sigma_{g_0} \right)^{\frac{4}{q+3}}}{a \left( \int_M |u|^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{\frac{n-2}{n}} + 2(n-1)b \left( \int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{\frac{n-2}{n-1}}}.$$

Hence the second assertion follows by Hölder's inequality and letting  $q \nearrow \frac{n+2}{n-2}$ .  $\square$

**Remark 3.2.** We point out that there also holds  $\lim_{q \nearrow \frac{n+2}{n-2}} \mu_q = Y_{a,b}(M, \partial M)$  when  $Q(M, \partial M)$  is a negative real number (cf. [15, Remark 7.1]).

Again thanks to [15], it is enough to prove Theorem 1.1 when  $Y(M, \partial M) \geq 0$ .

**Lemma 3.3.** *Let  $(M, g_0)$  be a smooth compact Riemannian manifold of dimension  $n \geq 3$ . Let  $2 \leq p < \frac{2(n-1)}{n-2}$ , then given any  $\epsilon > 0$ , there exists  $C = C(n, M, g_0) > 0$  such that*

$$\left( \int_{\partial M} |\varphi|^p d\sigma_{g_0} \right)^{\frac{2}{p}} \leq \epsilon \int_M |\nabla \varphi|_{g_0}^2 d\mu_{g_0} + \frac{C}{\epsilon} \int_M \varphi^2 d\mu_{g_0}$$

for any  $\varphi \in H^1(M, g_0)$ .

*Proof.* By negation, there exist some  $\epsilon_0 > 0$  and  $\{\varphi_j; j \in \mathbb{N}\} \subset H^1(M, g_0)$  such that

$$1 = \left( \int_{\partial M} |\varphi_j|^p d\sigma_{g_0} \right)^{\frac{2}{p}} > \epsilon_0 \int_M |\nabla \varphi_j|_{g_0}^2 d\mu_{g_0} + \frac{j}{\epsilon_0} \int_M \varphi_j^2 d\mu_{g_0}.$$

From this,  $\{\varphi_j\}$  is uniformly bounded in  $H^1(M, g_0)$  and  $\int_M \varphi_j^2 d\mu_{g_0} \rightarrow 0$  as  $j \rightarrow \infty$ . Then up to a subsequence,  $\varphi_j \rightharpoonup \varphi$  weakly in  $H^1(M, g_0)$ ,  $\varphi_j \rightarrow \varphi$  strongly in  $L^2(M, g_0)$  and  $L^p(\partial M, g_0)$  as  $j \rightarrow \infty$ . Notice that  $\varphi_j \rightarrow 0$  in  $L^2(M, g_0)$  as  $j \rightarrow \infty$ . Thus we obtain  $\varphi = 0$  a.e. in  $M$ , which contradicts  $\int_{\partial M} |\varphi|^p d\sigma_{g_0} = \lim_{j \rightarrow \infty} \int_{\partial M} |\varphi_j|^p d\sigma_{g_0} = 1$ .  $\square$

**Lemma 3.4.** *Let  $(M, g_0)$  be a smooth compact Riemannian manifold of dimension  $n \geq 3$  with boundary. Given  $a, b > 0$ , then*

(i) *Let  $\varphi \in C_c^\infty(\overline{\mathbb{R}_+^n})$ , there holds*

$$\begin{aligned} & a \left( \int_{\mathbb{R}_+^n} |\varphi|^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{n}} + 2(n-1)b \left( \int_{\mathbb{R}^{n-1}} |\varphi|^{\frac{2(n-1)}{n-2}} d\sigma \right)^{\frac{n-2}{n-1}} \\ & \leq \frac{1}{Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})} \frac{4(n-1)}{n-2} \int_{\mathbb{R}_+^n} |\nabla \varphi|^2 dy, \end{aligned}$$

*equality holds if and only if  $\varphi(y) = W(y)$  up to dilations and translations in variables  $y^1, \dots, y^{n-1}$ .*

(ii) *Suppose  $\varphi$  is a smooth function with compact support in a coordinate neighborhood  $B_\rho(x_0) \cap \bar{M}$ , then  $\forall \epsilon > 0$  there exists  $\rho_0$  such that  $\rho \in (0, \rho_0)$ ,*

$$\begin{aligned} & a \left( \int_M |\varphi|^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{\frac{n-2}{n}} + 2(n-1)b \left( \int_{\partial M} |\varphi|^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{\frac{n-2}{n-1}} \\ & \leq \frac{1+\epsilon}{Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})} \frac{4(n-1)}{n-2} \int_M |\nabla \varphi|_{g_0}^2 d\mu_{g_0}, \end{aligned}$$

where  $\rho_0$  is independent of  $x_0$ .

(iii) *Given  $\epsilon > 0$ , there exists  $C(\epsilon)$  such that for every  $\varphi \in H^1(M, g_0)$*

$$\begin{aligned} & a \left( \int_M |\varphi|^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{\frac{n-2}{n}} + 2(n-1)b \left( \int_{\partial M} |\varphi|^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{\frac{n-2}{n-1}} \\ & \leq \frac{1+\epsilon}{Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})} \frac{4(n-1)}{n-2} \int_M |\nabla \varphi|_{g_0}^2 d\mu_{g_0} + C(\epsilon) \int_M \varphi^2 d\mu_{g_0}. \end{aligned}$$

*Proof.* Assertion (i) is a direct consequence of [22, Theorem 3.3] and [25, Theorem 1.2]. Indeed (ii) and (iii) can be proved by a cut-and-paste argument.

(ii) Note that  $g_0$  is Euclidean in  $B_\rho(x_0) \cap \bar{M}$  up to order two under normal coordinates around  $x_0 \in M$  or order one under the Fermi coordinates around  $x_0 \in \partial M$ . Then the inequality follows from (i) for every  $\varphi$  compactly supported in this coordinate chart.

(iii) Choose a finite covering of  $\bar{M}$  by local coordinate charts, each of which satisfies the condition of part (ii). Through an argument of a partition of unity subordinate to this covering, the desired Sobolev inequality follows (e.g. [7]).  $\square$

**Lemma 3.5.** *For any  $1 < q < \frac{n+2}{n-2}$ , there exists a positive smooth minimizer  $u_q$  for  $\mu_q$ .*

*Proof.* Let  $\{u_i\} \subset H^1(M, g_0)$  be a minimizing sequence of nonnegative functions for  $\mu_q$  with the normalization:

$$a \left( \int_M u_i^{q+1} d\mu_{g_0} \right)^{\frac{2}{q+1}} + 2(n-1)b \left( \int_{\partial M} u_i^{\frac{q+3}{2}} d\sigma_{g_0} \right)^{\frac{4}{q+3}} = 1, \forall i \in \mathbb{N}.$$

It is routine to show  $u_i$  is uniformly bounded in  $H^1(M, g_0)$ . Up to a subsequence,  $u_i \rightharpoonup u_q$  in  $H^1(M, g_0)$  and  $u_i \rightarrow u_q$  in  $L^{q+1}(M, g_0)$  and  $L^{(q+3)/2}(\partial M, g_0)$  as  $i \rightarrow \infty$ . Thus we obtain

$$a \left( \int_M u_q^{q+1} d\mu_{g_0} \right)^{\frac{2}{q+1}} + 2(n-1)b \left( \int_{\partial M} u_q^{\frac{q+3}{2}} d\sigma_{g_0} \right)^{\frac{4}{q+3}} = 1. \quad (3.1)$$

Then it follows from Lemma 3.3 and (3.1) that

$$\int_M u_q^{q+1} d\mu_{g_0} \geq C_0 > 0. \quad (3.2)$$

Next we claim that  $\int_{\partial M} u_q^{(q+3)/2} d\sigma_{g_0} > 0$ . By contradiction, if  $u_q = 0$  a.e. on  $\partial M$ , namely  $u_q \in H_0^1(M, g_0)$ , then it yields

$$\mu_q = E[u_q] = \inf_{0 \neq v \in H_0^1(M, g_0)} \frac{E[v]}{a \left( \int_M |v|^{q+1} d\mu_{g_0} \right)^{\frac{2}{q+1}}}.$$

Thus the nonnegative minimizer  $u_q \in H_0^1(M, g_0)$  weakly solves

$$\begin{cases} -\frac{4(n-1)}{n-2} \Delta_{g_0} v + R_{g_0} v = \mu_q a^{\frac{q+1}{2}} v^q & \text{in } M, \\ \frac{\partial v}{\partial \nu_{g_0}} = 0 & \text{on } \partial M. \end{cases}$$

Hence a contradiction is reached by using Hopf boundary point lemma and (3.2).

Consequently  $u_q$  is a nonzero, nonnegative minimizer with normalization (3.1) for  $\mu_q$ . Then  $u_q \in H^1(M, g_0)$  weakly solves

$$\begin{cases} -\frac{4(n-1)}{n-2} \Delta_{g_0} u_q + R_{g_0} u_q = \mu_q a \left( \int_M u_q^{q+1} d\mu_{g_0} \right)^{\frac{1-q}{1+q}} u_q^q & \text{in } M, \\ \frac{2}{n-2} \frac{\partial u_q}{\partial \nu_{g_0}} + h_{g_0} u_q = \mu_q b \left( \int_{\partial M} u_q^{q+1} d\sigma_{g_0} \right)^{\frac{1-q}{q+3}} u_q^{\frac{q+1}{2}} & \text{on } \partial M. \end{cases} \quad (3.3)$$

Then the strong maximum principle gives  $u_q > 0$  in  $\bar{M}$ . Furthermore, a regularity theorem in [17] shows  $u_q$  is smooth in  $\bar{M}$ .  $\square$

**Proof of Theorem 1.1.** From Lemma 3.5 that for each  $1 < q < \frac{n+2}{n-2}$ , there exists a positive minimizer  $u_q \in H^1(M, g_0)$  with the normalization (3.1), which solves (3.3), namely for all  $\psi \in H^1(M, g_0)$ ,

$$\begin{aligned} & \int_M \left( \frac{4(n-1)}{n-2} \langle \nabla u_q, \nabla \psi \rangle_{g_0} + R_{g_0} u_q \psi \right) d\mu_{g_0} + 2(n-1) \int_{\partial M} h_{g_0} u_q \psi d\sigma_{g_0} \\ & - \mu_q \left[ a\alpha_q \int_M u_q^q \psi d\mu_{g_0} + 2(n-1)b\beta_q \int_{\partial M} u_q^{\frac{q+1}{2}} \psi d\sigma_{g_0} \right] = 0, \end{aligned} \quad (3.4)$$

where  $\alpha_q = \left( \int_M u_q^{q+1} d\mu_{g_0} \right)^{(1-q)/(1+q)}$ ,  $\beta_q = \left( \int_{\partial M} u_q^{(q+3)/2} d\sigma_{g_0} \right)^{(1-q)/(q+3)}$ . It follows from Lemma 3.1 and (3.1) that  $u_q$  is uniformly bounded in  $H^1(M, g_0)$ . Up to a subsequence,  $u_q$  weakly converges to some nonnegative function  $u$  in  $H^1(M, g_0)$  as  $q \nearrow \frac{n+2}{n-2}$ , and  $u$  weakly solves (2.3). Meanwhile, by Lemma 3.1 we get  $\mu_q \rightarrow Y_{a,b}(M, \partial M)$  as  $q \nearrow \frac{n+2}{n-2}$ .

From Lemma 3.4, for any  $\epsilon > 0$  there exists  $C(\epsilon) > 0$  such that

$$\begin{aligned} & a \left( \int_M u_q^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{\frac{n-2}{n}} + 2(n-1)b \left( \int_{\partial M} u_q^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{\frac{n-2}{n-1}} \\ & \leq (Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})^{-1} + \epsilon) \frac{4(n-1)}{n-2} \int_M |\nabla u_q|_{g_0}^2 d\mu_{g_0} + C(\epsilon) \int_M u_q^2 d\mu_{g_0}. \end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned} & \left( \int_M d\mu_{g_0} \right)^{\frac{n-2}{n} - \frac{2}{q+1}} \left( \int_M u_q^{q+1} d\mu_{g_0} \right)^{\frac{2}{q+1}} \leq \left( \int_M u_q^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{\frac{n-2}{n}}, \\ & \left( \int_{\partial M} d\sigma_{g_0} \right)^{\frac{n-2}{n-1} - \frac{4}{q+3}} \left( \int_{\partial M} u_q^{\frac{q+3}{2}} d\sigma_{g_0} \right)^{\frac{4}{q+3}} \leq \left( \int_{\partial M} u_q^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{\frac{n-2}{n-1}}. \end{aligned}$$

By choosing  $q$  sufficiently close to  $\frac{n+2}{n-2}$  and using the normalization (3.1), we get

$$\begin{aligned} & 1 - \epsilon \\ & \leq (Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})^{-1} + \epsilon) \frac{4(n-1)}{n-2} \int_M |\nabla u_q|_{g_0}^2 d\mu_{g_0} + C(\epsilon) \int_M u_q^2 d\mu_{g_0} \\ & = (Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})^{-1} + \epsilon) \left( \mu_q - \int_M R_{g_0} u_q^2 d\mu_{g_0} - 2(n-1) \int_{\partial M} h_{g_0} u_q^2 d\sigma_{g_0} \right) \\ & \quad + C(\epsilon) \int_M u_q^2 d\mu_{g_0} \\ & \leq (Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})^{-1} + 2\epsilon) Y_{a,b}(M, \partial M) + C \int_M u_q^2 d\mu_{g_0}, \end{aligned}$$

where the last inequality follows from Lemmas 3.4 and 3.3. By choosing  $\epsilon$  small enough and the assumption  $Y_{a,b}(M, \partial M) < Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})$ , we get

$$\int_M u_q^2 d\mu_{g_0} \geq C_1 > 0,$$

where  $C_1$  is independent of  $q$ . So  $\alpha_q$  is uniformly bounded, then after passing to a subsequence we let  $\bar{\alpha} = \lim_{q \nearrow \frac{n+2}{n-2}} \alpha_q > 0$ . Meanwhile using  $u_q \rightarrow u$  in  $L^2(M, g_0)$  as  $q \nearrow \frac{n+2}{n-2}$ , we obtain

$$\int_M u^2 d\mu_{g_0} > 0. \quad (3.5)$$

Next we claim that with a constant  $C_2$  independent of  $q$ , there holds

$$\int_{\partial M} u_q^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \geq C_2 > 0.$$

By negation, there exists a sequence  $\{u_q\}$  such that

$$\lim_{q \nearrow \frac{n+2}{n-2}} \int_{\partial M} u_q^{\frac{q+3}{2}} d\sigma_{g_0} = 0,$$

then we obtain  $\int_{\partial M} u^2 d\sigma_{g_0} = \lim_{q \nearrow \frac{n+2}{n-2}} \int_{\partial M} u_q^2 d\sigma_{g_0} = 0$ , which implies  $u = 0$  a. e. on  $\partial M$ . On the other hand, for any  $\psi \in H^1(M, g_0)$ , we get

$$\beta_q \left| \int_{\partial M} u_q^{\frac{q+1}{2}} \psi d\sigma_{g_0} \right| \leq \left( \int_{\partial M} u_q^{\frac{q+3}{2}} d\sigma_{g_0} \right)^{\frac{2}{q+3}} \|\psi\|_{L^{\frac{q+3}{2}}(M, g_0)} \rightarrow 0,$$

as  $q \rightarrow \infty$ . By letting  $q \nearrow \frac{n+2}{n-2}$  in equation (3.4),  $u$  weakly solves

$$\begin{cases} -\frac{4(n-1)}{n-2} \Delta_{g_0} u + R_{g_0} u = a\bar{\alpha} Y_{a,b}(M, \partial M) u^{\frac{n+2}{n-2}} & \text{in } M, \\ \frac{\partial u}{\partial \nu_{g_0}} + \frac{n-2}{2} h_{g_0} u = 0 & \text{on } \partial M. \end{cases}$$

From (3.5), Hopf boundary point lemma gives  $u > 0$  on  $\partial M$ . Hence we reach a contradiction.

Consequently, after passing to a further subsequence, we let  $0 < \bar{\beta} = \lim_{q \nearrow \frac{n+2}{n-2}} \beta_q$ . Furthermore, Fatou's lemma gives

$$\bar{\alpha} \leq \left( \int_M u^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{-\frac{2}{n}}, \quad \bar{\beta} \leq \left( \int_{\partial M} u^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{-\frac{1}{n-1}}.$$

Letting  $q \nearrow \frac{n+2}{n-2}$  in (3.4), we obtain

$$\int_M \left( \frac{4(n-1)}{n-2} \langle \nabla u, \nabla \psi \rangle_{g_0} + R_{g_0} u \psi \right) d\mu_{g_0} + 2(n-1) \int_{\partial M} h_{g_0} u \psi d\sigma_{g_0}$$

$$-Y_{a,b}(M, \partial M) \left[ a\bar{\alpha} \int_M u^{\frac{n+2}{n-2}} \psi d\mu_{g_0} + 2(n-1)b\bar{\beta} \int_{\partial M} u^{\frac{n}{n-2}} \psi d\sigma_{g_0} \right] = 0, \quad (3.6)$$

for all  $\psi \in H^1(M, g_0)$ . The strong maximum principle gives  $u > 0$  in  $\bar{M}$ . Test (3.6) with  $u$ , it yields

$$\begin{aligned} & Y_{a,b}(M, \partial M) \\ & \leq \mathcal{Q}_{a,b}[u] = \frac{Y_{a,b}(M, \partial M) \left[ a\bar{\alpha} \int_M u^{\frac{2n}{n-2}} d\mu_{g_0} + 2(n-1)b\bar{\beta} \int_{\partial M} u^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right]}{a \left( \int_M u^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{\frac{n-2}{n}} + 2(n-1)b \left( \int_{\partial M} u^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{\frac{n-2}{n-1}}} \\ & \leq Y_{a,b}(M, \partial M). \end{aligned}$$

From this, we conclude that

$$\bar{\alpha} = \left( \int_M u^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{-\frac{2}{n}}, \quad \bar{\beta} = \left( \int_{\partial M} u^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{-\frac{1}{n-1}}$$

and  $Y_{a,b}(M, \partial M) = \mathcal{Q}_{a,b}[u] = E[u]$ . Then  $u_q \rightarrow u$  in  $H^1(M, g_0)$  as  $q \nearrow \frac{n+2}{n-2}$  and  $u$  is a positive minimizer for  $Y_{a,b}(M, \partial M)$  and weakly solves (2.3). The regularity of  $u$  can follow from a theorem by Cherrier [17].  $\square$

## 4 Compactness of minimizers for various (a,b)

For brevity, we denote by  $u_{a,b}$  the positive smooth minimizer of  $Y_{a,b}(M, \partial M)$  with the normalization

$$a \left( \int_M u_{a,b}^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{\frac{n-2}{n}} + 2(n-1)b \left( \int_{\partial M} u_{a,b}^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{\frac{n-2}{n-1}} = 1. \quad (4.1)$$

Under the conformal change of  $g = u_{a,b}^{4/(n-2)} g_0$ , we have

$$R_g = aY_{a,b}(M, \partial M) \left( \int_M u_{a,b}^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{-\frac{2}{n}}$$

and

$$h_g = bY_{a,b}(M, \partial M) \left( \int_{\partial M} u_{a,b}^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{-\frac{1}{n-1}}.$$

Modulo a positive constant multiple, we get  $R_g = 1$  and

$$h_g = \frac{b}{\sqrt{a}} \sqrt{Y_{a,b}(M, \partial M)} \left( \int_M u_{a,b}^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{\frac{1}{n}} \left( \int_{\partial M} u_{a,b}^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{-\frac{1}{n-1}}. \quad (4.2)$$

Let  $K$  be a compact set of  $\{(a, b); a \geq 0, b \geq 0\} \setminus \{(0, 0)\}$ .

**Proposition 4.1.** Assume  $Y(M, \partial M) \geq 0$  and let  $(a, b) \in K$ , then  $Y_{a,b}(M, \partial M)$  is non-increasing in  $a$  for any fixed  $b$ , as well as in  $b$  for any fixed  $a$ , and is continuous in  $K$ .

*Proof.* The proof is in the spirit of that of [20, Proposition 3.2]. For simplicity, we only prove the assertions for  $a$  with fixed  $b$ , the others are similar. Notice that  $Y(M, \partial M) \geq 0$ , then  $E[u] \geq 0$  for any  $u \in H^1(M, g_0)$ . For  $0 \leq a_1 \leq a_2$ ,  $Y_{a_1,b}(M, \partial M) \geq Y_{a_2,b}(M, \partial M)$  follows from

$$\mathcal{Q}_{a_1,b}[u] \geq \mathcal{Q}_{a_2,b}[u] \text{ for any } u \in H^1(M, g_0).$$

Next we prove the continuity of  $Y_{a,b}(M, \partial M)$  in  $K$ . Since  $Y(M, \partial M) \geq 0$  we may assume the background metric  $g_0$  satisfies  $R_{g_0} = 0$  in  $M$  and  $h_{g_0} \geq 0$  on  $\partial M$ . Suppose  $\{(a_m, b_m); m \in \mathbb{N}\} \subset K$  and  $(a_m, b_m) \rightarrow (a, b) \in K$  as  $m \rightarrow \infty$ . We assume  $a \geq 0, b > 0$  for simplicity. On one hand, given any  $\epsilon > 0$ , there exists a  $u \in H^1(M, g_0) \setminus \{0\}$  such that  $\mathcal{Q}_{a,b}[u] < Y_{a,b}(M, \partial M) + \epsilon$ . For this fixed  $u$ ,  $\mathcal{Q}_{a_m,b_m}[u] \rightarrow \mathcal{Q}_{a,b}[u]$  as  $m \rightarrow \infty$ . Then

$$\lim_{m \rightarrow \infty} Y_{a_m,b_m}(M, \partial M) \leq \lim_{m \rightarrow \infty} \mathcal{Q}_{a_m,b_m}[u] = \mathcal{Q}_{a,b}[u] < Y_{a,b}(M, \partial M) + \epsilon.$$

On the other hand, given any  $\epsilon > 0$ , for each  $(a_m, b_m)$  there exists  $u_m \in H^1(M, g_0)$  with

$$a_m \left( \int_M u_m^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{\frac{n-2}{n}} + 2(n-1)b_m \left( \int_{\partial M} u_m^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{\frac{n-2}{n-1}} = 1$$

such that  $E[u_m] < Y_{a_m,b_m}(M, \partial M) + \epsilon$ .

Let  $0 \leq a_0 = \inf_m a_m$  and  $0 < b_0 = \inf_m b_m$ . Then it follows from the monotonicity of  $Y_{a,b}(M, \partial M)$  that

$$Y_{a_m,b_m}(M, \partial M) \leq Y_{a_m,b_0}(M, \partial M) \leq Y_{a_0,b_0}(M, \partial M).$$

From the above normalization of  $u_m$ , we get

$$\begin{aligned} \frac{4(n-1)}{n-2} \int_M |\nabla u_m|_{g_0}^2 d\mu_{g_0} &= E[u_m] - 2(n-1) \int_{\partial M} h_{g_0} u_m^2 d\sigma_{g_0} \\ &\leq Y_{a_0,b_0}(M, \partial M) + \epsilon + C \int_{\partial M} u_m^2 d\sigma_{g_0} \leq Y_{a_0,b_0}(M, \partial M) + C. \end{aligned}$$

This yields  $\{u_m\}$  is uniformly bounded in  $H^1(M, g_0)$ . Thus for all sufficiently large  $m$ , we have

$$a \left( \int_M u_m^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{\frac{n-2}{n}} + 2(n-1)b \left( \int_{\partial M} u_m^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{\frac{n-2}{n-1}} > 1 - \epsilon.$$

Consequently, we obtain

$$Y_{a,b}(M, \partial M) \leq \mathcal{Q}_{a,b}[u_m] < \frac{E[u_m]}{1 - \epsilon} < \frac{Y_{a_m,b_m}(M, \partial M) + \epsilon}{1 - \epsilon}$$

for all sufficiently large  $m$ . □



**Lemma 4.2.** Suppose  $Y_{a,b}(M, \partial M) < Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})$  for all  $(a, b) \in K$ . Let  $u_{a,b}$  be any positive smooth minimizer for  $Y_{a,b}(M, \partial M)$  satisfying the normalization (4.1), then there exists  $C = C(K, g_0) > 0$  such that

$$\int_M u_{a,b}^2 d\mu_{g_0} \geq C, \quad \int_{\partial M} u_{a,b}^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \geq C, \quad \forall (a, b) \in K.$$

*Proof.* For  $a = 0$ , the desired assertions are guaranteed by [19, Proposition 2.1]. So in the following we assume  $a > 0$ . Given any  $\epsilon > 0$ , by Lemma 3.4 it yields

$$\begin{aligned} & Y_{a,b}(M, \partial M)^{-1} E[u_{a,b}] \\ &= a \left( \int_M u_{a,b}^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{\frac{n-2}{n}} + 2(n-1)b \left( \int_{\partial M} u_{a,b}^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{\frac{n-2}{n-1}} \\ &\leq (Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})^{-1} + \epsilon) \frac{4(n-1)}{n-2} \int_M |\nabla u_{a,b}|_{g_0}^2 d\mu_{g_0} + C(\epsilon) \int_M u_{a,b}^2 d\mu_{g_0}. \end{aligned}$$

Since  $Y(M, \partial M) \geq 0$ , we choose an initial metric such that  $R_{g_0} \geq 0$  and  $h_{g_0} \geq 0$ . From Proposition 4.1 that  $Y_{a,b}(M, \partial M)$  is continuous in  $K$ , then there exists  $k_0 > 0$  such that

$$\min_K \{Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1}) - Y_{a,b}(M, \partial M)\} \geq k_0.$$

By choosing  $\epsilon$  sufficiently small, with a constant  $C = C(k_0) > 0$  we obtain

$$\int_M |\nabla u_{a,b}|_{g_0}^2 d\mu_{g_0} \leq C \int_M u_{a,b}^2 d\mu_{g_0}. \quad (4.3)$$

First we claim that  $\forall (a, b) \in K$ ,  $\int_M u_{a,b}^2 d\mu_{g_0} \geq \bar{C}_1(K, g_0) > 0$ . Otherwise there exists a sequence of minimizers  $u_m := u_{a_m, b_m}$  with  $(a_m, b_m) \in K$  such that  $\int_M u_m^2 d\mu_{g_0} \rightarrow 0$ , then (4.3) gives  $\|u_m\|_{H^1(M, g_0)} \rightarrow 0$  as  $m \rightarrow \infty$ , which contradicts the normalization (4.1) of  $u_m$ .

Next we assert that  $\forall (a, b) \in K$ ,  $\int_{\partial M} u_{a,b}^{2(n-1)/(n-2)} d\sigma_{g_0} \geq C(K, g_0) > 0$ . By negation, there exists a sequence of minimizers  $u_m = u_{a_m, b_m}$  with  $(a_m, b_m) \in K$  such that  $\int_{\partial M} u_m^{2(n-1)/(n-2)} d\sigma_{g_0} \rightarrow 0$  as  $m \rightarrow \infty$ . Since  $K$  is compact, we may assume  $(a_m, b_m) \rightarrow (a, b) \in K$  and  $E[u_m] = Y_{a_m, b_m}(M, \partial M) \rightarrow Y_{a,b}(M, \partial M)$  by Proposition 4.1 as  $m \rightarrow \infty$ . Notice that  $E[u_m] = Y_{a_m, b_m}(M, \partial M)$ , it follows from Proposition 4.1 that  $u_m$  is uniformly bounded in  $H^1(M, g_0)$ . Up to a subsequence, there hold  $u_m \rightharpoonup u$  weakly in  $H^1(M, g_0)$  and

$$\begin{aligned} \int_M u^2 d\mu_{g_0} &= \lim_{m \rightarrow \infty} \int_M u_m^2 d\mu_{g_0} \geq \bar{C}_1, \\ \int_{\partial M} u^2 d\sigma_{g_0} &= \lim_{m \rightarrow \infty} \int_{\partial M} u_m^2 d\sigma_{g_0} = 0. \end{aligned}$$

This means  $u \not\equiv 0$  and  $u = 0$  a. e. on  $\partial M$ . On the other hand,  $u_m$  satisfies

$$\int_M \left( \frac{4(n-1)}{n-2} \langle \nabla u_m, \nabla \psi \rangle_{g_0} + R_{g_0} u_m \psi \right) d\mu_{g_0} + 2(n-1) \int_{\partial M} h_{g_0} u_m \psi d\sigma_{g_0}$$

$$\begin{aligned}
&= \left[ 2(n-1)b \left( \int_{\partial M} u_m^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{-\frac{1}{n-1}} \int_{\partial M} u_m^{\frac{n}{n-2}} \psi d\sigma_{g_0} \right. \\
&\quad \left. + a \left( \int_M u_m^{\frac{2n}{n-2}} d\sigma_{g_0} \right)^{-\frac{2}{n}} \int_M u_m^{\frac{n+2}{n-2}} \psi d\sigma_{g_0} \right] Y_{a,b,m}(M, \partial M)
\end{aligned} \tag{4.4}$$

for all  $\psi \in H^1(M, g_0)$ . By Hölder's inequality and the normalization (4.1) for  $u_m$ , we have

$$\left( \int_{\partial M} u_m^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{-\frac{1}{n-1}} \int_{\partial M} u_m^{\frac{n}{n-2}} \psi d\sigma_{g_0} \rightarrow 0$$

and

$$\int_M u_m^{\frac{2n}{n-2}} d\mu_{g_0} \rightarrow a^{\frac{n}{2-n}}, \text{ as } m \rightarrow \infty.$$

By letting  $m \rightarrow \infty$  in (4.4),  $u$  weakly solves

$$\begin{cases} -\frac{4(n-1)}{n-2} \Delta_{g_0} u + R_{g_0} u = a^{\frac{n}{n-2}} Y_{a,b}(M, \partial M) u^{\frac{n+2}{n-2}} & \text{in } M, \\ \frac{2}{n-2} \frac{\partial u}{\partial \nu_{g_0}} + h_{g_0} u = 0 & \text{on } \partial M. \end{cases}$$

Then Hopf boundary point lemma gives  $u > 0$  on  $\partial M$ . This yields a contradiction.  $\square$

Based on these preparations, we are now in a position to establish Theorem 1.5.

**Proof of Theorem 1.5.** We only need proof the assertion for  $Y(M, \partial M) \geq 0$  due to the same reason of [15].

First we claim that there exists  $C = C(K, g_0)$  such that  $u_{a,b} \leq C$  for any  $(a, b) \in K$ . By contradiction, suppose there exist sequences  $\{(a_m, b_m); m \in \mathbb{N}\} \subset K$  and  $\{p_m; m \in \mathbb{N}\} \subset \bar{M}$  such that

$$r_m := u_{a_m, b_m}(p_m) = \max_{x \in \bar{M}} u_{a_m, b_m}(x) \rightarrow \infty \text{ as } m \rightarrow \infty.$$

For brevity, we set  $u_m = u_{a_m, b_m}$ . Since  $M$  is compact, we may assume  $p_m \rightarrow p_0 \in \bar{M}$  as  $m \rightarrow \infty$ .

If  $\lim_{m \rightarrow \infty} \text{dist}_{g_0}(p_m, \partial M) r_m^{\frac{2}{n-2}} = \infty$ , under normal coordinates around  $p_0$ , near  $p_0$  there holds

$$(g_0)_{ij}(x) = \delta_{ij} + O(|x|^2).$$

Observe that

$$\frac{4(n-1)}{n-2} \frac{1}{\sqrt{\det g_0}} \partial_i (\sqrt{\det g_0} g_0^{ij} \partial_j u_m) - R_{g_0} u_m + \tilde{a}_m u_m^{\frac{n+2}{n-2}} = 0$$

in  $\Omega_\rho$ , where

$$\tilde{a}_m = a_m \left( \int_M u_m^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{-\frac{2}{n}} Y_{a_m, b_m}(M, \partial M).$$

Define  $\rho_m = \rho r_m^{\frac{2}{n-2}}$  and

$$v_m(y) = r_m^{-1} u_m(\exp_{p_m}(y r_m^{-\frac{2}{n-2}})) \text{ for } y \in B_{\rho_m}(0) \subset \mathbb{R}^n.$$

Then  $v_m(0) = 1$  and  $0 < v_m(y) \leq 1$  in  $B_{\rho_m}(0)$ . Let  $g_m(y) = g_0(\exp_{p_m}(y r_m^{-\frac{2}{n-2}}))$ ,  $f_m(y) = r_m^{-\frac{4}{n-2}} R_{g_0}(\exp_{p_m}(y r_m^{-\frac{2}{n-2}}))$ . Then  $v_m$  satisfies

$$\frac{4(n-1)}{n-2} \frac{1}{\sqrt{\det g_m}} \frac{\partial}{\partial y^i} (\sqrt{\det g_m} g_m^{ij} \frac{\partial}{\partial y^j} v_m) - f_m v_m + \tilde{a}_m v_m^{\frac{n+2}{n-2}} = 0$$

in  $B_{\rho_m}(0)$ . As  $m \rightarrow \infty$ , there hold

$$(g_m)_{ij} \rightarrow \delta_{ij} \quad f_m \rightarrow 0 \text{ in } C^1(\hat{K}) \text{ for any compact set } \hat{K} \subset \mathbb{R}^n.$$

Since  $K$  is compact and from Lemma 4.2 that  $\tilde{a}_m$  is uniformly bounded, up to a subsequence we get

$$(a_m, b_m) \rightarrow (a, b), \quad \tilde{a}_m \rightarrow \tilde{a}, \text{ as } m \rightarrow \infty.$$

From the  $W^{2,p}$ -estimate,  $\|v_m\|_{C^\lambda(B_{r_m})}$  is uniformly bounded for any  $\lambda \in (0, 1)$ . Applying Schauder interior estimates and the diagonal method to extract a subsequence from  $\{v_m\}$ , still denote as  $\{v_m\}$ , we obtain  $v_m \rightarrow v$  in  $C^{2,\lambda}(\hat{K})$ , as  $m \rightarrow \infty$ . Moreover  $v$  satisfies

$$\frac{4(n-1)}{n-2} \Delta v + \tilde{a} v^{\frac{n-2}{n+2}} = 0 \text{ in } \mathbb{R}^n.$$

Notice that  $v(0) = 1$  and  $0 \leq v \leq 1$ , the strong maximum principle gives  $v > 0$ . From Fatou's lemma, we have

$$\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} dx \leq \liminf_{m \rightarrow \infty} \int_{B_{\rho_m}(0)} v_m^{\frac{2n}{n-2}} \sqrt{\det g_m} dx \leq \liminf_{m \rightarrow \infty} \int_M u_m^{\frac{2n}{n-2}} d\mu_{g_0}. \quad (4.5)$$

Recall that

$$\tilde{a} = a Y_{a,b}(M, \partial M) \lim_{m \rightarrow \infty} \left( \int_M u_m^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{-\frac{2}{n}}.$$

It is not hard to show that if  $Y(M, \partial M) = 0$ , then  $Y_{a,b}(M, \partial M) = 0$  for any  $(a, b) \in K$ . If either  $a = 0$  or  $Y(M, \partial M) = 0$ , then  $\tilde{a} = 0$ . Then the strong maximum principle gives  $v \equiv 1$ . Using similar arguments in Lemma 4.2, one can show that  $u_m$  is uniformly bounded in  $H^1(M, g_0)$ . From this and (4.5), we have

$$\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} dx \leq C,$$

which contradicts  $v \equiv 1$  in  $\mathbb{R}^n$ . If  $Y(M, \partial M) > 0$  and  $a > 0$ , then  $\tilde{a} > 0$ . Observe that

$$\tilde{a} \int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} dx = \frac{4(n-1)}{n-2} \int_{\mathbb{R}^n} |\nabla v|^2 dx = 2^{\frac{2}{n}} a Y_{a,0}(\mathbb{R}_+^n, \mathbb{R}^{n-1}) \left( \int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}. \quad (4.6)$$

Together with Proposition 4.1, (4.5) and (4.6) give

$$Y_{a,0}(\mathbb{R}_+^n, \mathbb{R}^{n-1}) \geq Y_{a,0}(M, \partial M) \geq Y_{a,b}(M, \partial M) \geq 2^{\frac{2}{n}} Y_{a,0}(\mathbb{R}_+^n, \mathbb{R}^{n-1}),$$

which obviously yields a contradiction.

If  $\lim_{m \rightarrow \infty} \text{dist}_{g_0}(p_m, \partial M) r_m^{\frac{2}{n-2}} < \infty$ . Let  $X = (x^1, \dots, x^{n-1})$  be the normal coordinates of  $x \in \partial M$  around  $p_0$  and  $\nu(X) := \nu_{g_0}(X)$  be the unit outward normal at  $x \in \partial M$ . For small  $t \geq 0$ ,  $\exp_X(-t\nu(X)) : B_\rho^+(0) \rightarrow \Omega_\rho \subset M$  is a diffeomorphism, then  $(x^1, \dots, x^{n-1}, t)$  are called the Fermi coordinates around  $p_0$ . Without loss of generality, we assume  $p_m \in \Omega_\rho$  and denote by  $p_m = \exp_{X_m}(-t_m \nu(X_m))$ .

Under these coordinates, we have

$$\begin{cases} \frac{4(n-1)}{n-2} \frac{1}{\sqrt{\det g_0}} \partial_i (\sqrt{\det g_0} g_0^{ij} \partial_j u_m) - R_{g_0} u_m + \tilde{a}_m u_m^{\frac{n+2}{n-2}} = 0 & \text{in } \Omega_\rho, \\ \frac{2}{n-2} \frac{\partial u_m}{\partial \nu_{g_0}} + h_{g_0} u_m = \tilde{b}_m u_m^{\frac{n}{n-2}} & \text{on } \partial\Omega_\rho \cap \partial M, \end{cases}$$

where

$$\begin{aligned} \tilde{a}_m &= a_m \left( \int_M u_m^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{-\frac{2}{n}} Y_{a_m, b_m}(M, \partial M), \\ \tilde{b}_m &= b_m \left( \int_{\partial M} u_m^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{-\frac{1}{n-1}} Y_{a_m, b_m}(M, \partial M). \end{aligned}$$

Define  $\rho_m = \rho r_m^{\frac{2}{n-2}}$  and

$$v_m(X, t) = r_m^{-1} u_m(\exp_{X_m}(-t_m r_m^{-\frac{2}{n-2}} \nu(X r_m^{-\frac{2}{n-2}}))) \text{ in } B_{\rho_m}^+(0).$$

Then  $v_m(0) = 1$  and  $0 < v_m(X, t) \leq 1$  in  $B_{\rho_m}^+(0)$ . We set

$$\begin{aligned} g_m(X, t) &= g_0(\exp_{X_m}(-t_m r_m^{-\frac{2}{n-2}} \nu(X r_m^{-\frac{2}{n-2}}))), \\ \tilde{f}_m(X, t) &= r_m^{-\frac{4}{n-2}} R_{g_0}(\exp_{X_m}(-t_m r_m^{-\frac{2}{n-2}} \nu(X r_m^{-\frac{2}{n-2}}))), \\ h_m(X) &= r_m^{-\frac{2}{n-2}} h_{g_0}(X r_m^{-\frac{2}{n-2}}). \end{aligned}$$

Thus  $v_m$  satisfies

$$\begin{cases} \frac{4(n-1)}{n-2} \frac{1}{\sqrt{\det g_m}} \partial_i (\sqrt{\det g_m} g_m^{ij} \partial_j v_m) - \tilde{f}_m v_m + \tilde{a}_m v_m^{\frac{n+2}{n-2}} = 0 & \text{in } B_{\rho_m}^+, \\ -\frac{2}{n-2} \partial_t v_m + h_m v_m - \tilde{b}_m v_m^{\frac{n}{n-2}} = 0 & \text{on } D_{\rho_m}. \end{cases}$$

Since  $r_m \rightarrow \infty$  as  $m \rightarrow \infty$ , we have

$$(g_m)_{ij} \rightarrow \delta_{ij}, \quad \tilde{f}_m, h_m \rightarrow 0 \text{ in } C^1(\tilde{K})$$

for any compact set  $\tilde{K} \subset \overline{\mathbb{R}_+^n}$ . Since  $K$  is compact and from Lemma 4.2 that  $\tilde{a}_m, \tilde{b}_m$  are bounded, up to a subsequence we have

$$(a_m, b_m) \rightarrow (a, b), \quad \tilde{a}_m \rightarrow \tilde{a}, \quad \tilde{b}_m \rightarrow \tilde{b} \quad \text{as } m \rightarrow \infty.$$

From  $W^{2,p}$ -estimate,  $\|v_m\|_{C^\lambda(\overline{B_{\rho_m}^+})}$  is uniformly bounded for any  $\lambda \in (0, 1)$ . Applying Schauder estimates and the diagonal method to extract a subsequence from  $\{v_m\}$ , still denote as  $\{v_m\}$ , we obtain  $v_m \rightarrow v$  in  $C^{2,\lambda}(\tilde{K})$ , as  $m \rightarrow \infty$ . Moreover  $v$  satisfies

$$\begin{cases} \frac{4(n-1)}{n-2} \Delta v + \tilde{a} v^{\frac{n-2}{n+2}} = 0 & \text{in } \mathbb{R}_+^n, \\ -\frac{\partial v}{\partial t} - \tilde{b} v^{\frac{n}{n-2}} = 0 & \text{on } \mathbb{R}^{n-1}. \end{cases} \quad (4.7)$$

Notice that  $v(0) = 1$  and  $0 \leq v \leq 1$ , the strong maximum principle gives  $v > 0$ . Recall that

$$\begin{aligned} \tilde{a} &= a Y_{a,b}(M, \partial M) \lim_{m \rightarrow \infty} \left( \int_M u_m^{\frac{2n}{n-2}} d\mu_{g_0} \right)^{-\frac{2}{n}}, \\ \tilde{b} &= b Y_{a,b}(M, \partial M) \lim_{m \rightarrow \infty} \left( \int_{\partial M} u_m^{\frac{2(n-1)}{n-2}} d\sigma_{g_0} \right)^{-\frac{1}{n-1}}. \end{aligned}$$

Fatou's lemma gives

$$\begin{aligned} \int_{\mathbb{R}_+^n} v^{\frac{2n}{n-2}} dx &\leq \liminf_{m \rightarrow \infty} \int_{B_{\rho_m}^+} v_m^{\frac{2n}{n-2}} \sqrt{\det g_m} dx \leq \liminf_{m \rightarrow \infty} \int_M u_m^{\frac{2n}{n-2}} d\mu_{g_0}, \\ \int_{\mathbb{R}^{n-1}} v^{\frac{2(n-1)}{n-2}} d\sigma &\leq \liminf_{m \rightarrow \infty} \int_{D_{\rho_m}} v_m^{\frac{2(n-1)}{n-2}} \sqrt{\det g_m} d\sigma \leq \liminf_{m \rightarrow \infty} \int_{\partial M} u_m^{\frac{2(n-1)}{n-2}} d\sigma_{g_0}. \end{aligned}$$

If  $Y(M, \partial M) = 0$ , then  $\tilde{a} = \tilde{b} = 0$ . Then the strong maximum principle gives  $v \equiv 1$  in  $\mathbb{R}_+^n$ . As above, we also get  $v \in L^{2n/(n-2)}(\mathbb{R}_+^n)$ . Thus we reach a contradiction. If  $Y(M, \partial M) > 0$ , testing with  $v$  in problem (4.7), we get

$$\begin{aligned} &\tilde{a} \int_{\mathbb{R}_+^n} v^{\frac{2n}{n-2}} dx + 2(n-1) \tilde{b} \int_{\mathbb{R}^{n-1}} v^{\frac{2(n-1)}{n-2}} d\sigma \\ &= \frac{4(n-1)}{n-2} \int_{\mathbb{R}_+^n} |\nabla v|^2 dx \\ &= \left[ a \left( \int_{\mathbb{R}_+^n} v^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} + 2(n-1)b \left( \int_{\mathbb{R}^{n-1}} v^{\frac{2(n-1)}{n-2}} d\sigma \right)^{\frac{n-2}{n-1}} \right] Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1}). \end{aligned}$$

Since  $a^2 + b^2 > 0$  and  $Y(M, \partial M) > 0$  imply  $\tilde{a}^2 + \tilde{b}^2 > 0$ , we have

$$Y_{a,b}(M, \partial M) = \lim_{m \rightarrow \infty} Y_{a_m, b_m}(M, \partial M) \geq Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1}),$$

which contradicts the assumption  $Y_{a,b}(M, \partial M) < Y_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})$ ,  $\forall (a, b) \in K$ .

Finally based on the above upper bound, it follows from Lemma 4.2 and [2, Proposition A-4] that  $\forall (a, b) \in K$ ,  $u_{a,b}$  has a positive uniform lower bound. Then Schauder estimates give the  $C^2$ -estimate of  $u_{a,b}$  in  $K$ .  $\square$

As a byproduct of Proposition 4.1, there hold

$$\begin{aligned}\lim_{a \rightarrow 0^+} Y_{a,b}(M, \partial M) &= Y_{0,b}(M, \partial M), \quad \text{for any fixed } b > 0, \\ \lim_{b \rightarrow 0^+} Y_{a,b}(M, \partial M) &= Y_{a,0}(M, \partial M), \quad \text{for any fixed } a > 0.\end{aligned}$$

From these together with Theorem 1.5, when  $Y(M, \partial M) > 0$  expression (4.2) shows that the normalized conformal metric of scalar curvature 1 has positive constant mean curvature, which runs in a large set of  $\mathbb{R}_+$ .

## 5 Construction of test functions

In this section, we use the following notation: given any  $\rho > 0$ , let

$$\begin{aligned}B_\rho^+(0) &= B_\rho(0) \cap \mathbb{R}_+^n; \quad \partial^+ B_\rho^+(0) = \partial B_\rho^+(0) \cap \mathbb{R}_+^n; \\ D_\rho(0) &= \partial B_\rho^+(0) \setminus \partial^+ B_\rho^+(0).\end{aligned}$$

From now on, we assume  $Y(M, \partial M) > 0$ . Recall that  $d = [(n-2)/2]$  when  $n \geq 3$ . By the result of Marques [28], for each  $x_0 \in \partial M$  there exists a conformal metric  $g_{x_0} = f_{x_0}^{4/(n-2)} g_0$  with  $f_{x_0}(x_0) = 1$ . Suppose  $\Psi_{x_0} : B_{2\rho}^+(0) \rightarrow M$  is the  $g_{x_0}$ -Fermi coordinates around  $x_0$ , set  $x = \Psi_{x_0}(y)$  for  $y \in B_{2\rho}^+(0)$ . Under these coordinates, there hold  $\det g_{x_0} = 1 + O(|y|^{2d+2})$ ,  $(g_{x_0})_{ij}(0) = \delta_{ij}$  and  $(g_{x_0})_{ni}(y) = \delta_{ni}$ , for any  $y \in B_{2\rho}^+(0)$  and  $i, j = 1, \dots, n$ . Let  $g_{x_0} = \exp(h)$ , where  $\exp$  denotes the matrix exponential, then the symmetric 2-tensor  $h$  has the following properties:

$$\begin{cases} \operatorname{tr} h(y) = O(|y|^{2d+2}), & \text{for } y \in B_{2\rho}^+(0), \\ h_{ab}(0) = 0, & \text{for } i, j = 1, \dots, n, \\ h_{in}(y) = 0, & \text{for } y \in B_{2\rho}^+(0), i = 1, \dots, n, \\ \partial_a h_{bc}(0) = 0, & \text{for } a, b, c = 1, \dots, n-1, \\ \sum_{b=1}^{n-1} y^b h_{ab}(y) = 0, & \text{for } y \in D_{2\rho}(0), a = 1, \dots, n-1. \end{cases} \quad (5.1)$$

The last two properties follow from the fact that Fermi coordinates are normal on  $\partial M$ .

**Convention.** In the following, we let  $a, b, c, \dots$  range from 1 to  $n-1$  and  $i, j, k, \dots$  range from 1 to  $n$ . We adopt Einstein summation convention and simplify  $B_\rho^+(0), \partial^+ B_\rho^+(0), D_\rho(0)$  by  $B_\rho^+, \partial^+ B_\rho^+, D_\rho$  without otherwise stated.

Under these conformal Fermi coordinates, the mean curvature satisfies

$$\begin{aligned}h_{g_{x_0}}(x) &= -\frac{1}{2(n-1)} g^{ab} \partial_n g_{ab}(x) \\ &= -\frac{1}{2(n-1)} \partial_n (\log \det(g_{x_0}))(x) = O(|y|^{2d+1}).\end{aligned} \quad (5.2)$$

Let  $H_{ij}$  be the Taylor expansion of  $h_{ij}$  up to order  $d$ , namely

$$H_{ij} = \sum_{|\alpha|=1}^d \partial^\alpha h_{ij} y^\alpha,$$

where  $\alpha$  is a multi-index and  $\partial^\alpha h_{ij} = \partial^\alpha h_{ij}(0)$ . Then  $H$  satisfies (5.1) except the first property replaced by  $\text{tr}H = 0$ .

## 5.1 Linearization of scalar curvature and mean curvature

From (2.2) and (2.1), we get

$$W_\epsilon \partial_i \partial_j W_\epsilon - \frac{n}{n-2} \partial_i W_\epsilon \partial_j W_\epsilon = \frac{1}{n} \left( W_\epsilon \Delta W_\epsilon - \frac{n}{n-2} |\nabla W_\epsilon|^2 \right) \delta_{ij} \text{ in } \mathbb{R}_+^n. \quad (5.3)$$

**Proposition 5.1.** *Let  $V$  be a smooth vector field in  $\overline{\mathbb{R}_+^n}$  satisfying  $V_n = 0 = \partial_n V_a$  on  $\mathbb{R}^{n-1}$ , where  $1 \leq a \leq n-1$ . Let*

$$\psi = V_k \partial_k W_\epsilon + \frac{n-2}{2n} W_\epsilon \text{div} V$$

and

$$S_{ij} = \partial_i V_j + \partial_j V_i - \frac{2}{n} \text{div} V \delta_{ij}$$

be a conformal killing operator. Then we have

$$\Delta \psi + n(n+2) W_\epsilon^{\frac{4}{n-2}} \psi = \frac{n-2}{4(n-1)} W_\epsilon \partial_i \partial_j S_{ij} + \partial_i (\partial_j W_\epsilon S_{ij}) \text{ in } \mathbb{R}_+^n \quad (5.4)$$

and

$$\partial_n \psi - \frac{n}{n-2} W_\epsilon^{-1} \partial_n W_\epsilon \psi = \frac{1}{2} \partial_n W_\epsilon S_{nn} + \frac{n-2}{4(n-1)} W_\epsilon \partial_n S_{nn} \text{ on } \mathbb{R}^{n-1}. \quad (5.5)$$

*Proof.* The linearized equations (5.4) and (5.5) for scalar curvature and mean curvature can be verified by direct computations in [11, Proposition 5] and [13, Proposition 5], respectively. Somewhat inspired by Brendle [11], we adopt a geometric proof of these linearized equations. It involves the first variation formulae for scalar curvature and mean curvature at a round metric of the spherical cap  $\Sigma$ .

Let  $g_\Sigma = W_\epsilon^{4/(n-2)} g_{\mathbb{R}^n}$  be the standard spherical metric on  $\Sigma$  of constant sectional curvature 4, see also Section 2. We now consider a family of perturbed metrics of  $g_\Sigma$ :

$$W_\epsilon^{\frac{4}{n-2}} e^{tS} = \phi_t^* ((W_\epsilon - t\psi)^{\frac{4}{n-2}} g_{\mathbb{R}^n}), \quad t \in \mathbb{R}, \quad (5.6)$$

where  $\phi_t$  is one-parameter family of diffeomorphisms on  $S^n$  generated by  $V$ . Differentiating of (5.6) with respect to  $t$  and evaluating at  $t = 0$ , we get

$$W_\epsilon^{\frac{4}{n-2}} S = \mathcal{L}_V(g_\Sigma) - \frac{4}{n-2} \psi W_\epsilon^{-1} g_\Sigma. \quad (5.7)$$

We remark that such a decomposition of symmetric 2-tensor is guaranteed by [8, Lemma 4.57]. Recall that the first variation of scalar curvature (cf. [8, Theorem 1.174 (e)]) is given by:

$$R'_g(h) = -h^{ik}R_{ik} + \nabla^i \nabla^k h_{ik} - \Delta_g \text{tr}_g(h) \quad (5.8)$$

for any symmetric 2-tensor  $h$ , where  $\nabla$  indicates the covariant derivative of  $g$ .

On one hand, set  $\tilde{g}_E = e^{tS}$ , there holds

$$R_{W_\epsilon^{\frac{4}{n-2}} \tilde{g}_E} = W_\epsilon^{-\frac{n+2}{n-2}} \left( -\frac{4(n-1)}{n-2} \Delta_{\tilde{g}_E} W_\epsilon + R_{\tilde{g}_E} W_\epsilon \right).$$

Notice that  $\det \tilde{g}_E = 1$  due to  $\text{tr} S = 0$ , then

$$\frac{d}{dt} \Big|_{t=0} \Delta_{\tilde{g}_E} W_\epsilon = \frac{d}{dt} \Big|_{t=0} \partial_i (e^{-tS_{ij}} \partial_j W_\epsilon) = -\partial_i (S_{ij} \partial_j W_\epsilon)$$

and (5.8) gives

$$\frac{d}{dt} \Big|_{t=0} R_{\tilde{g}_E} = \partial_i \partial_j S_{ij}.$$

Thus we obtain

$$\begin{aligned} R'_{g_\Sigma}(W_\epsilon^{\frac{4}{n-2}} S) &= \frac{d}{dt} \Big|_{t=0} R_{W_\epsilon^{\frac{4}{n-2}} \tilde{g}_E} \\ &= W_\epsilon^{-\frac{n+2}{n-2}} \left( \frac{4(n-1)}{n-2} \partial_i (S_{ij} \partial_j W_\epsilon) + \partial_i \partial_j S_{ij} W_\epsilon \right). \end{aligned} \quad (5.9)$$

On the other hand, using (5.7) and (5.8), we have

$$R'_{g_\Sigma}(W_\epsilon^{\frac{4}{n-2}} S) = R'_{g_\Sigma}(\mathcal{L}_V(g_\Sigma)) - R'_{g_\Sigma}\left(\frac{4}{n-2} \psi W_\epsilon^{-1} g_\Sigma\right), \quad (5.10)$$

where  $\mathcal{L}_V(g_\Sigma)$  denotes the Lie derivative of metric  $g_\Sigma$  along the vector field  $V$ . In particular, it is routine to verify that

$$R'_{g_\Sigma}(\mathcal{L}_V(g_\Sigma)) = 0. \quad (5.11)$$

It also follows from (5.8) that

$$\begin{aligned} &R'_{g_\Sigma}\left(\frac{4}{n-2} \psi W_\epsilon^{-1} g_\Sigma\right) \\ &= \frac{4}{n-2} \left[ -4n(n-1) W_\epsilon^{-1} \psi + (1-n) \Delta_{g_\Sigma}(W_\epsilon^{-1} \psi) \right] \\ &= -\frac{4(n-1)}{n-2} \left[ 4n W_\epsilon^{-1} \psi + n(n-2) W_\epsilon^{-1} \psi + W_\epsilon^{-\frac{n+2}{n-2}} \Delta \psi \right] \\ &= -\frac{4(n-1)}{n-2} W_\epsilon^{-\frac{n+2}{n-2}} \left[ n(n+2) W_\epsilon^{\frac{4}{n-2}} \psi + \Delta \psi \right]. \end{aligned} \quad (5.12)$$

Putting (5.9)-(5.12) together, we obtain equation (5.4).



Next we need to show (5.5). Let  $\nu_g$  be the unit outward normal on  $\mathbb{R}^{n-1}$ , then

$$\nu_g = -\frac{g^{ni}}{\sqrt{g^{nn}}}\partial_i$$

and

$$\begin{aligned} h_g &= -\frac{1}{n-1}g^{ab}\langle \nu_g, \nabla_{\partial_a}\partial_b \rangle = \frac{1}{n-1}g^{ab}g^{ni}(g^{nn})^{-\frac{1}{2}}g_{ij}\Gamma_{ab}^j \\ &= \frac{1}{n-1}g^{ab}\Gamma_{ab}^n(g^{nn})^{-\frac{1}{2}}. \end{aligned} \quad (5.13)$$

From conformal change formula of mean curvatures, we get

$$\left. \frac{d}{dt} \right|_{t=0} h_{W_\epsilon^{\frac{4}{n-2}}\tilde{g}_E} = \frac{2}{n-2}W_\epsilon^{-\frac{n}{n-2}} \left. \frac{d}{dt} \right|_{t=0} \left( \frac{\partial W_\epsilon}{\partial \nu_{\tilde{g}_E}} + \frac{n-2}{2}h_{\tilde{g}_E}W_\epsilon \right). \quad (5.14)$$

Observe that

$$\frac{\partial W_\epsilon}{\partial \nu_{\tilde{g}_E}} = -(\tilde{g}_E^{nn})^{-1/2}\tilde{g}_E^{ni}\partial_i W_\epsilon,$$

then

$$\left. \frac{d}{dt} \right|_{t=0} \frac{\partial W_\epsilon}{\partial \nu_{\tilde{g}_E}} = S_{ni}\partial_i W_\epsilon - \frac{1}{2}S_{nn}\partial_n W_\epsilon = \frac{1}{2}S_{nn}\partial_n W_\epsilon, \quad (5.15)$$

where the last identity follows from  $S_{an} = 0$  on  $\mathbb{R}^{n-1}$  due to the assumption that  $V_n = 0 = \partial_n V_a$  on  $\mathbb{R}^{n-1}$ . Recall that the Christoffel symbols of  $\tilde{g}_E$  are given by

$$\tilde{\Gamma}_{ab}^n = \frac{1}{2}\tilde{g}_E^{ni}[\partial_b(\tilde{g}_E)_{ai} + \partial_a(\tilde{g}_E)_{ib} - \partial_i(\tilde{g}_E)_{ab}]$$

then

$$\left. \frac{d}{dt} \right|_{t=0} \tilde{\Gamma}_{ab}^n = -\frac{1}{2}\partial_n S_{ab},$$

due to  $S_{an} = 0$  on  $\mathbb{R}^{n-1}$ . From this and (5.13), we get

$$\left. \frac{d}{dt} \right|_{t=0} h_{\tilde{g}_E} = -\frac{1}{2(n-1)}\partial_n S_{aa} = \frac{1}{2(n-1)}\partial_n S_{nn}, \quad (5.16)$$

where the last identity follows from  $-S_{nn} = S_{aa}$  due to  $\text{tr}S = 0$ . Plugging (5.15) and (5.16) into (5.14), we obtain

$$\left. \frac{d}{dt} \right|_{t=0} h_{W_\epsilon^{\frac{4}{n-2}}\tilde{g}_E} = \frac{2}{n-2}W_\epsilon^{-\frac{n}{n-2}} \left( \frac{1}{2}\partial_n W_\epsilon S_{nn} + \frac{n-2}{4(n-1)}W_\epsilon\partial_n S_{nn} \right). \quad (5.17)$$

On the other hand, using (5.7) we have

$$h'_{g_\Sigma}(W_\epsilon^{\frac{4}{n-2}}S) = h'_{g_\Sigma}(\mathcal{L}_V(g_\Sigma)) - h'_{g_\Sigma}\left(\frac{4}{n-2}\psi W_\epsilon^{-1}g_\Sigma\right) \text{ on } \mathbb{R}^{n-1}. \quad (5.18)$$

First we assert that

$$h'_{g_\Sigma}(\mathcal{L}_V(g_\Sigma)) = 0 \quad \text{on } \mathbb{R}^{n-1}. \quad (5.19)$$

Next we compute

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} h_{(W_\epsilon - t\psi)^{\frac{4}{n-2}} g_E} &= -\frac{2}{n-2} \frac{d}{dt} \Big|_{t=0} \left[ (W_\epsilon - t\psi)^{-\frac{n}{n-2}} \partial_n (W_\epsilon - t\psi) \right] \\ &= \frac{2}{n-2} W_\epsilon^{-\frac{n}{n-2}} \left( \partial_n \psi - \frac{n}{n-2} W_\epsilon^{-1} \partial_n W_\epsilon \psi \right). \end{aligned} \quad (5.20)$$

Therefore from (5.17)-(5.20), equation (5.5) follows.

It remains to show assertion (5.19). Define

$$\hat{S}_{ij} := \mathcal{L}_V(g_\Sigma)_{ij} = (V_k \partial_k W_\epsilon^{\frac{4}{n-2}}) \delta_{ij} + W_\epsilon^{\frac{4}{n-2}} (\partial_i V_j + \partial_j V_i).$$

For brevity, we abuse  $g = g_\Sigma$  for a while. Since  $V_n = 0 = \partial_n V_a$  on  $\mathbb{R}_+^n$ , then  $\hat{S}_{an} = 0$  on  $\mathbb{R}_+^n$ . Observe that

$$(\Gamma_{ab}^n)' = \frac{1}{2} g^{ni} (\nabla_b \hat{S}_{ia} + \nabla_a \hat{S}_{ib} - \nabla_i \hat{S}_{ab}) = \frac{1}{2} W_\epsilon^{-\frac{4}{n-2}} (\nabla_b \hat{S}_{na} + \nabla_a \hat{S}_{nb} - \nabla_n \hat{S}_{ab}),$$

then

$$\begin{aligned} g^{ab} (\Gamma_{ab}^n)' &= W_\epsilon^{-\frac{4}{n-2}} \left[ g^{ab} \hat{S}_{na,b} - \frac{1}{2} \partial_n \text{tr}_g(\hat{S}) \right] \\ &= W_\epsilon^{-\frac{4}{n-2}} \left[ W_\epsilon^{-\frac{4}{n-2}} \hat{S}_{na,a} - \frac{1}{2} \partial_n (W_\epsilon^{-\frac{4}{n-2}} \hat{S}_{aa}) \right]. \end{aligned}$$

We compute

$$\begin{aligned} \hat{S}_{na,a} &= \partial_a \hat{S}_{na} - \Gamma_{na}^i \hat{S}_{ia} - \Gamma_{aa}^i \hat{S}_{ni} \\ &= -\Gamma_{na}^b \hat{S}_{ba} - \Gamma_{aa}^n \hat{S}_{nn} \\ &= -2T_c W_\epsilon^{\frac{2}{n-2}} [\hat{S}_{aa} - (n-1)\hat{S}_{nn}], \end{aligned}$$

where the last identity follows from

$$\begin{aligned} \Gamma_{na}^b &= \frac{1}{2} g^{bc} \partial_n g_{ca} = \frac{1}{2} W_\epsilon^{-\frac{4}{n-2}} \partial_n W_\epsilon^{\frac{4}{n-2}} \delta_{ab} = 2T_c W_\epsilon^{\frac{2}{n-2}} \delta_{ab}, \\ \Gamma_{aa}^n &= -\frac{1}{2} g^{nn} \partial_n g_{aa} = -\frac{1}{2} W_\epsilon^{-\frac{4}{n-2}} \partial_n W_\epsilon^{\frac{4}{n-2}} \delta_{aa} = -2(n-1)T_c W_\epsilon^{\frac{2}{n-2}} \end{aligned}$$

in view of (2.1). From (5.13), we have

$$(n-1)(h_g)'(\hat{S}) = -\hat{S}^{ab} \pi_{ab} + \frac{n-1}{2} \frac{\hat{S}^{nn}}{g^{nn}} h_g + \frac{(\Gamma_{ab}^n)'}{\sqrt{g^{nn}}} g^{ab}.$$

From (5.3) we get

$$\partial_n \partial_a W_\epsilon^{\frac{4}{n-2}} = \frac{4}{n-2} \left[ \frac{6-n}{n-2} W_\epsilon^{\frac{4}{n-2}-2} \partial_n W_\epsilon \partial_a W_\epsilon + \partial_n \partial_a W_\epsilon \right]$$

$$= \frac{4}{n-2} \frac{6}{n-2} W_\epsilon^{\frac{4}{n-2}-2} \partial_n W_\epsilon \partial_a W_\epsilon = 6T_c W_\epsilon^{\frac{2}{n-2}} \partial_a W_\epsilon^{\frac{4}{n-2}},$$

whence

$$\begin{aligned} \partial_n \hat{S}_{aa} &= \partial_n \left[ (n-1)(V_k \partial_k W_\epsilon^{\frac{4}{n-2}}) + 2W_\epsilon^{\frac{4}{n-2}} \partial_a V_a \right] \\ &= (n-1)(V_a \partial_n \partial_a W_\epsilon^{\frac{4}{n-2}} + \partial_n V_n \partial_n W_\epsilon^{\frac{4}{n-2}}) + 2\partial_n W_\epsilon^{\frac{4}{n-2}} \partial_a V_a \\ &= (n-1)T_c W_\epsilon^{\frac{2}{n-2}} (6V_a \partial_a W_\epsilon^{\frac{4}{n-2}} + 4\partial_n V_n W_\epsilon^{\frac{4}{n-2}}) + 8T_c W_\epsilon^{\frac{6}{n-2}} \partial_a V_a. \end{aligned}$$

Then we have

$$\begin{aligned} \partial_n (W_\epsilon^{-\frac{4}{n-2}} \hat{S}_{aa}) &= W_\epsilon^{-\frac{4}{n-2}} \partial_n \hat{S}_{aa} + \partial_n W_\epsilon^{-\frac{4}{n-2}} \hat{S}_{aa} \\ &= (n-1)T_c W_\epsilon^{-\frac{2}{n-2}} (6V_a \partial_a W_\epsilon^{\frac{4}{n-2}} + 4\partial_n V_n W_\epsilon^{\frac{4}{n-2}}) \\ &\quad + 8T_c W_\epsilon^{\frac{2}{n-2}} \partial_a V_a - 4T_c W_\epsilon^{-\frac{2}{n-2}} \hat{S}_{aa}. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} &-T_c^{-1} W_\epsilon^{\frac{6}{n-2}} g^{ab} (\Gamma_{ab}^n)' \\ &= 2[\hat{S}_{aa} - (n-1)\hat{S}_{nn}] + (n-1)(3V_a \partial_a W_\epsilon^{\frac{4}{n-2}} + 2\partial_n V_n W_\epsilon^{\frac{4}{n-2}}) \\ &\quad + 4W_\epsilon^{\frac{4}{n-2}} \partial_a V_a - 2\hat{S}_{aa} \\ &= -2(n-1)\hat{S}_{nn} + 4W_\epsilon^{\frac{4}{n-2}} \partial_a V_a + (n-1)(3V_a \partial_a W_\epsilon^{\frac{4}{n-2}} + 2\partial_n V_n W_\epsilon^{\frac{4}{n-2}}). \end{aligned}$$

Putting these facts together and using  $\pi_{ab} = -2T_c g_{ab}$ , we conclude that

$$\begin{aligned} &(n-1)T_c^{-1} W_\epsilon^{\frac{4}{n-2}} (h_g)'(\hat{S}) \\ &= 2\hat{S}_{aa} - (n-1)\hat{S}_{nn} + T_c^{-1} W_\epsilon^{\frac{6}{n-2}} g^{ab} (\Gamma_{ab}^n)' \\ &= 2\hat{S}_{aa} + (n-1)\hat{S}_{nn} - (n-1)(3V_a \partial_a W_\epsilon^{\frac{4}{n-2}} + 2\partial_n V_n W_\epsilon^{\frac{4}{n-2}}) - 4W_\epsilon^{\frac{4}{n-2}} \partial_a V_a \\ &= 2 \left[ (n-1)(V_a \partial_a W_\epsilon^{\frac{4}{n-2}}) + 2W_\epsilon^{\frac{4}{n-2}} \partial_a V_a \right] \\ &\quad + (n-1) \left[ (V_a \partial_a W_\epsilon^{\frac{4}{n-2}}) + 2W_\epsilon^{\frac{4}{n-2}} \partial_n V_n \right] \\ &\quad - (n-1)(3V_a \partial_a W_\epsilon^{\frac{4}{n-2}} + 2\partial_n V_n W_\epsilon^{\frac{4}{n-2}}) - 4W_\epsilon^{\frac{4}{n-2}} \partial_a V_a \\ &= 0, \end{aligned}$$

which implies the desired assertion.  $\square$

## 5.2 Test functions and their energy estimates

Let  $\chi(y) = \chi(|y|)$  be a smooth cut-off function in  $\overline{\mathbb{R}_+^n}$  with  $\chi = 1$  in  $B_1^+$  and  $\chi = 0$  in  $\overline{\mathbb{R}_+^n} \setminus B_2^+$ . For any  $\rho > 0$ , set  $\chi_\rho(y) = \chi(|y|/\rho)$  for  $y \in \mathbb{R}_+^n$ . As in [12] and [13], given  $H_{ij}$

there exists a smooth vector field  $V$  in  $\overline{\mathbb{R}_+^n}$  such that

$$\begin{cases} \sum_{i=1}^n \partial_i \left[ W_\epsilon^{\frac{2n}{n-2}} \left( \chi_\rho H_{ij} - \partial_i V_j - \partial_j V_i + \frac{2}{n} (\operatorname{div} V) \delta_{ij} \right) \right] = 0, & \text{in } \mathbb{R}_+^n, \\ \partial_n V_a = V_n = 0, & \text{on } \mathbb{R}^{n-1}, \end{cases} \quad (5.21)$$

where  $1 \leq i, j \leq n, 1 \leq a \leq n-1$ . Moreover, there holds

$$|\partial^\beta V(y)| \leq C(n, |\beta|) \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| (\epsilon + |y|)^{|\alpha|+1-|\beta|}. \quad (5.22)$$

We only sketch the proof of the construction of vector field  $V$ . Consider the spherical cap  $(\Sigma, g_\Sigma)$  as in Proposition 5.1 with  $\epsilon = 1$ . Define

$$\mathcal{X} = \{V \in H^1(\Sigma, g_\Sigma); \langle V, \nu_{g_\Sigma} \rangle_{g_\Sigma} = 0 \text{ for a vector field } V \text{ on } \partial\Sigma\}$$

and  $\mathcal{H}$  the space of all trace-free symmetric two-tensors on  $\Sigma$  of class  $L^2$ . A conformal killing operator  $\mathcal{D} : \mathcal{X} \rightarrow \mathcal{H}$  on  $\Sigma$  defined as

$$\mathcal{D}_{g_\Sigma} V = \mathcal{L}_V(g_\Sigma) - \frac{2}{n} (\operatorname{div}_{g_\Sigma} V) g_\Sigma.$$

Similarly as in the appendix of [12], we know that  $\ker \mathcal{D}_{g_\Sigma}$  is finite dimensional. We define

$$\mathcal{X}_0 = \{V \in \mathcal{X}; \langle V, Z \rangle_{L^2(\Sigma, g_\Sigma)} = 0, \forall Z \in \ker \mathcal{D}_{g_\Sigma}\}.$$

Using a similar argument in [12, Proposition A.3], we assert that for any symmetric two-tensor  $\tilde{h}$  with compact support in  $\mathbb{R}_+^n$ , there exists a unique vector field  $V \in \mathcal{X}_0$  such that

$$\langle W^{\frac{4}{n-2}} \tilde{h} - \mathcal{D}_{g_\Sigma} V, \mathcal{D}_{g_\Sigma} Z \rangle_{L^2(\Sigma, g_\Sigma)} = 0 \text{ for all } Z \in \mathcal{X}.$$

Furthermore, with a dimensional constant  $C$  there holds

$$\|V\|_{L^2(\Sigma, g_\Sigma)}^2 + \|\nabla V\|_{L^2(\Sigma, g_\Sigma)}^2 \leq C \|W^{\frac{4}{n-2}} \tilde{h}\|_{L^2(\Sigma, g_\Sigma)}^2.$$

Based on this estimate and using our  $W$  instead, we can construct the vector field  $V$  satisfying (5.21) and estimate (5.22) by mimicking the proofs of [13, Propositions 12-13].

As in Proposition 5.1, we define symmetric trace-free 2-tensors  $S$  and  $T$  in  $\overline{\mathbb{R}_+^n}$  by

$$S_{ij} = \partial_i V_j + \partial_j V_i - \frac{2}{n} \operatorname{div} V \delta_{ij} \quad \text{and} \quad T = H - S. \quad (5.23)$$

It follows from (5.21) that  $T$  satisfies

$$W_\epsilon \partial_j T_{ij} + \frac{2n}{n-2} \partial_j W_\epsilon T_{ij} = 0, \quad \text{in } B_{2\rho}^+. \quad (5.24)$$

For  $n \geq 3$ , we define an auxiliary function  $\psi = \psi_{\epsilon, \rho, H}$  by

$$\psi = \partial_i W_\epsilon V_i + \frac{n-2}{2n} W_\epsilon \operatorname{div} V. \quad (5.25)$$

When  $n = 3$ , then  $d = 0$  and we choose  $\psi = 0$ . Using (5.22) and (2.2) of  $W_\epsilon$ , in  $B_{2\rho}^+$  we have

$$|\psi(y)| \leq C(n, T_c) \epsilon^{\frac{n-2}{2}} \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| (\epsilon + |y|)^{|\alpha|+2-n}. \quad (5.26)$$

By the above construction of  $V$  and  $H_{in} = 0$  in  $B_{2\rho}^+$ , we know

$$W_\epsilon \partial_i S_{ni} + \frac{2n}{n-2} \partial_i W_\epsilon S_{ni} = 0 \quad \text{in } B_{2\rho}^+$$

and  $S_{na} = 0$  on  $D_{2\rho}$ . Thus we get

$$\partial_n S_{nn} = -\partial_a S_{na} - \frac{2n}{n-2} W_\epsilon^{-1} \partial_i W_\epsilon S_{ni} = -\frac{2n}{n-2} W_\epsilon^{-1} \partial_n W_\epsilon S_{nn} \quad \text{on } D_{2\rho}.$$

Combining this and (5.5), we conclude that

$$\partial_n \psi - \frac{n}{n-2} W_\epsilon^{-1} \partial_n W_\epsilon \psi = -\frac{1}{2(n-1)} \partial_n W_\epsilon S_{nn} \quad \text{on } D_{2\rho}.$$

For future citation, we collect the linearized equations for scalar curvature and mean curvature in the following

**Lemma 5.2.** *The function  $\psi$  satisfies*

$$\begin{cases} \Delta \psi + n(n+2) W_\epsilon^{\frac{4}{n-2}} \psi = \frac{n-2}{4(n-1)} W_\epsilon \partial_i \partial_j S_{ij} + \partial_i (\partial_j W_\epsilon S_{ij}) & \text{in } B_{2\rho}^+, \\ \partial_n \psi - \frac{n}{n-2} W_\epsilon^{-1} \partial_n W_\epsilon \psi = -\frac{1}{2(n-1)} \partial_n W_\epsilon S_{nn} & \text{on } D_{2\rho}. \end{cases}$$

Similar to [13, Proposition 5], we collect and derive some properties associated to  $S$  and  $T$ .

**Lemma 5.3.** (1)  $S_{an} = 0 = T_{an}$ ,  $0 \leq a \leq n-1$ .

(2) On  $D_{2\rho}$ , there hold

$$\begin{aligned} \partial_n S_{nn} &= -\frac{2n}{n-2} W_\epsilon^{-1} \partial_n W_\epsilon S_{nn}, \\ \partial_n S_{ab} &= -\frac{1}{n-1} \partial_n S_{nn} \delta_{ab}, \end{aligned}$$

where  $1 \leq a, b \leq n-1$ .

Based on Lemma 5.2, we rearrange [11, Propositions 5-6] as follows.

**Proposition 5.4.** *There holds*

$$\begin{aligned} & \frac{1}{4}Q_{ik,j}Q_{ik,j} - \frac{1}{2}Q_{ki,k}Q_{li,l} + 2W_\epsilon^{\frac{2n}{n-2}}T_{ik}T_{ik} \\ &= \frac{1}{4}W_\epsilon^2\partial_l H_{ik}\partial_l H_{ik} - \frac{2(n-1)}{n-2}\partial_k W_\epsilon\partial_l W_\epsilon H_{ik}H_{il} - 2W_\epsilon\partial_k W_\epsilon H_{ik}\partial_l H_{il} \\ & \quad - \frac{1}{2}W_\epsilon^2\partial_k H_{ik}\partial_l H_{il} + \frac{8(n-1)}{n-2}\partial_i W_\epsilon\partial_k \psi H_{ik} - \frac{4(n-1)}{n-2}|\nabla\psi|^2 \\ & \quad + \frac{4(n-1)}{n-2}n(n+2)W_\epsilon^{\frac{4}{n-2}}\psi^2 - 2W_\epsilon\psi\partial_i\partial_k H_{ik} + \operatorname{div}\xi, \end{aligned}$$

where

$$Q_{ij,k} = W_\epsilon\partial_k T_{ij} + \frac{2}{n-2}(\partial_l W_\epsilon T_{il}\delta_{jk} + \partial_l W_\epsilon T_{jl}\delta_{ik} - \partial_i W_\epsilon T_{jk} - \partial_j W_\epsilon T_{ik})$$

and the vector field  $\xi$  is given by

$$\begin{aligned} \xi_i &= 2W_\epsilon\psi\partial_k H_{ik} - 2W_\epsilon\partial_k \psi H_{ik} - 2\partial_k W_\epsilon\psi H_{ik} - \frac{1}{2}W_\epsilon^2\partial_i S_{lk}H_{lk} \\ & \quad + W_\epsilon^2\partial_l S_{kl}H_{ik} + 2W_\epsilon\partial_l W_\epsilon S_{kl}H_{ik} - W_\epsilon\psi\partial_k S_{ik} + W_\epsilon\partial_k \psi S_{ik} \\ & \quad + \partial_k W_\epsilon\psi S_{ik} + \frac{1}{4}W_\epsilon^2\partial_i S_{lk}S_{lk} - \frac{1}{2}W_\epsilon^2\partial_l S_{kl}S_{ik} - W_\epsilon\partial_l W_\epsilon S_{kl}S_{ik} \\ & \quad - \frac{4(n-1)}{n-2}\partial_k W_\epsilon\psi S_{ik} + \frac{4(n-1)}{n-2}\psi\partial_i\psi - \frac{2}{n-2}W_\epsilon\partial_k W_\epsilon T_{lk}T_{il}. \end{aligned} \quad (5.27)$$

In particular, it yields

$$\xi_n = -\frac{n+2}{2(n-2)}W_\epsilon\partial_n W_\epsilon S_{nn}^2 + \frac{4n(n-1)}{(n-2)^2}W_\epsilon^{-1}\partial_n W_\epsilon\psi^2 \quad (5.28)$$

on  $\mathbb{R}^{n-1}$ .

**Proposition 5.5.** *There exists  $\lambda^* = \lambda^*(n, T_c) > 0$  such that*

$$\begin{aligned} & \lambda^*\epsilon^{n-2}\sum_{i,j=1}^n\sum_{|\alpha|=1}^d|\partial^\alpha h_{ij}|^2\int_{B_\rho^+(0)}(\epsilon+|y|)^{2|\alpha|+2-2n}dy \\ & \leq \frac{1}{4}\int_{B_\rho^+(0)}Q_{ij,k}Q_{ij,k}dy - \frac{n^2}{2(n-1)(n-2)}\int_{D_\rho(0)}\partial_n W_\epsilon W_\epsilon S_{nn}^2 d\sigma \end{aligned}$$

for all  $2\epsilon \leq \rho$ .

*Proof.* Since only the unchanged sign condition of  $\partial_n W_\epsilon$  on  $B_\rho^+$  and Lemma 5.3 were required in [2, Lemma 3.4], we refer to similar arguments in [2, Proposition 3.5] for the details.  $\square$

Our test function is

$$\bar{U}_{(x_0, \epsilon)} = [\chi_\rho(W_\epsilon + \psi)] \circ \Psi_{x_0}^{-1} + (1 - \chi_\rho) \circ \Psi_{x_0}^{-1} \epsilon^{\frac{n-2}{2}} G, \quad (5.29)$$

where  $G = G_{x_0}$  is the Green's function of the conformal Laplacian with pole at  $x_0 \in \partial M$ , coupled with a boundary condition, namely

$$\begin{cases} -\frac{4(n-1)}{n-2} \Delta_{g_{x_0}} G_{x_0} + R_{g_{x_0}} G_{x_0} = 0, & \text{in } M \setminus \{x_0\}, \\ \frac{2}{n-2} \frac{\partial G_{x_0}}{\partial \nu_{g_{x_0}}} + h_{g_{x_0}} G_{x_0} = 0, & \text{on } \partial M \setminus \{x_0\}. \end{cases} \quad (5.30)$$

We assume that  $G$  is normalized such that  $\lim_{y \rightarrow 0} G(\Psi_{x_0}(y))|y|^{n-2} = 1$ . Then  $G$  satisfies the following estimates near  $x_0$ , namely for sufficiently small  $|y|$  (cf. [4, Proposition B-2]):

$$\begin{aligned} & |G(\Psi_{x_0}(y)) - |y|^{2-n}| \\ & \leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| |y|^{|\alpha|+2-n} + \begin{cases} C|y|^{d+3-n}, & \text{if } n \geq 5, \\ C(1 + |\log |y||), & \text{if } n = 3, 4, \end{cases} \\ & |\nabla(G(\Psi_{x_0}(y)) - |y|^{2-n})| \leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| |y|^{|\alpha|+1-n} + C|y|^{d+2-n}. \end{aligned} \quad (5.31)$$

Moreover, there holds

$$C(T_c, n)^{-1} \epsilon^{\frac{n-2}{2}} (\epsilon + |y|)^{2-n} \leq W_\epsilon(y) \leq C(T_c, n) \epsilon^{\frac{n-2}{2}} (\epsilon + |y|)^{2-n}.$$

We consider the flux integral as in [12, P.1006]

$$\begin{aligned} \mathcal{I}(x_0, \rho) = & - \int_{\partial^+ B_\rho^+} |y|^{2-2n} (|y|^2 \partial_j h_{ij} - 2n y^j h_{ij}) \frac{y^i}{|y|} d\sigma \\ & + \frac{4(n-1)}{n-2} \int_{\partial^+ B_\rho^+} (|y|^{2-n} \partial_i G - G \partial_i |y|^{2-n}) \frac{y^i}{|y|} d\sigma \end{aligned}$$

for  $x_0 \in \partial M$  and all sufficiently small  $\rho > 0$ .

The following estimates on the expansion of scalar curvature can be found in [2, P. 2645], which follows from [11, Proposition 11] and [13, Proposition 3]. Keep in mind that the boundary is not necessarily umbilic here.

**Proposition 5.6.** *The scalar curvature  $R_{g_{x_0}}$  satisfies*

$$\begin{aligned} |R_{g_{x_0}} - \partial_i \partial_k H_{ik}| & \leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| |y|^{|\alpha|-1} + C|y|^{d-1}, \\ \left| R_{g_{x_0}} - \partial_i \partial_k h_{ik} + \partial_k (H_{ik} \partial_l H_{il}) - \frac{1}{2} \partial_k H_{ik} \partial_l H_{il} + \frac{1}{4} \partial_l H_{ik} \partial_l H_{ik} \right| & \end{aligned}$$

$$\leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 |y|^{2|\alpha|-1} + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| |y|^{|\alpha|+d-1} + C |y|^{2d}$$

for  $|y|$  sufficiently small.

In order to prove this theorem, we need to estimate the energy  $E[\bar{U}_{(x_0,\epsilon)}]$ . Notice that

$$\begin{aligned} E[\bar{U}_{(x_0,\epsilon)}] &= \int_M \left( \frac{4(n-1)}{n-2} |\nabla \bar{U}_{(x_0,\epsilon)}|_{g_{x_0}}^2 + R_{g_{x_0}} \bar{U}_{(x_0,\epsilon)}^2 \right) d\mu_{g_{x_0}} \\ &\quad + 2(n-1) \int_{\partial M} h_{g_{x_0}} \bar{U}_{(x_0,\epsilon)}^2 d\sigma_{g_{x_0}}. \end{aligned}$$

We will estimate  $E[\bar{U}_{(x_0,\epsilon)}]$  in  $\Psi_{x_0}(B_\rho^+)$  and  $M \setminus \Psi_{x_0}(B_\rho^+)$  respectively.

**Proposition 5.7.** *With some sufficiently small  $\rho_0 > 0$ , there holds*

$$\begin{aligned} &\int_{B_\rho^+} \left[ \frac{4(n-1)}{n-2} |\nabla(W_\epsilon + \psi)|_{g_{x_0}}^2 + R_{g_{x_0}}(W_\epsilon + \psi)^2 \right] dy \\ &\quad + 2(n-1) \int_{D_\rho} h_{g_{x_0}}(W_\epsilon + \psi)^2 d\sigma \\ &\leq 4n(n-1) \int_{B_\rho^+} W_\epsilon^{\frac{4}{n-2}} \left( W_\epsilon^2 + \frac{n+2}{n-2} \psi^2 \right) dy \\ &\quad + \int_{\partial^+ B_\rho^+} \frac{4(n-1)}{n-2} \partial_i W_\epsilon W_\epsilon \frac{y^i}{|y|} d\sigma + \int_{\partial^+ B_\rho^+} (W_\epsilon^2 \partial_j h_{ij} - \partial_j W_\epsilon^2 h_{ij}) \frac{y^j}{|y|} d\sigma \\ &\quad - 4(n-1) T_c \int_{D_\rho} W_\epsilon^{\frac{2}{n-2}} \left( W_\epsilon^2 + 2W_\epsilon \psi + \frac{n}{n-2} \psi^2 - \frac{n-2}{8(n-1)^2} W_\epsilon^2 S_{nn}^2 \right) d\sigma \\ &\quad - \frac{1}{2} \lambda^* \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 \epsilon^{n-2} \int_{B_\rho^+} (\epsilon + |y|)^{2|\alpha|+2-2n} dy \\ &\quad + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \epsilon^{n-2} \rho^{|\alpha|+2-n} + C \epsilon^{n-2} \rho^{2d+4-n} \end{aligned}$$

for  $0 < 2\epsilon < \rho < \rho_0 \leq 1$ , where  $\rho_0$  and  $C$  are some constants depending only on  $n, T_c, g_0$ .

*Proof.* Notice that  $\bar{U}_{(x_0,\epsilon)} = W_\epsilon + \psi$  in  $B_\rho^+$ . First it follows from (5.2) and (5.25) that

$$\int_{D_\rho} h_{g_{x_0}}(W_\epsilon + \psi)^2 d\sigma \leq C \int_{D_\rho} |y|^{2d+1} (W_\epsilon + \psi)^2 d\sigma \leq C \epsilon^{n-2} \rho^{2d+2}. \quad (5.32)$$

Next we decompose

$$\frac{4(n-1)}{n-2} |\nabla(W_\epsilon + \psi)|_{g_{x_0}}^2 + R_{g_{x_0}}(W_\epsilon + \psi)^2$$



$$= \frac{4(n-1)}{n-2} |\nabla W_\epsilon|^2 + \frac{4(n-1)}{n-2} n(n+2) W_\epsilon^{\frac{4}{n-2}} \psi^2 + \sum_{i=1}^4 J_i, \quad (5.33)$$

where

$$\begin{aligned} J_1 &= \frac{8(n-1)}{n-2} \partial_i W_\epsilon \partial_i \psi - \frac{4(n-1)}{n-2} \partial_i W_\epsilon \partial_k W_\epsilon h_{ik} + W_\epsilon^2 \partial_i \partial_k h_{ik}, \\ &\quad - W_\epsilon^2 \partial_k (H_{ik} \partial_l H_{il}) - 2W_\epsilon \partial_k W_\epsilon H_{ik} \partial_l H_{il}, \\ J_2 &= -\frac{1}{4} W_\epsilon^2 \partial_l H_{ik} \partial_l H_{ik} + \frac{2(n-1)}{n-2} \partial_k W_\epsilon \partial_l W_\epsilon H_{ik} H_{il} + 2W_\epsilon \partial_k W_\epsilon H_{ik} \partial_l H_{il} \\ &\quad + \frac{1}{2} W_\epsilon^2 \partial_k H_{ik} \partial_l H_{il} + 2W_\epsilon \psi \partial_i \partial_k H_{ik} - \frac{8(n-1)}{n-2} \partial_i W_\epsilon \partial_k \psi H_{ik} \\ &\quad + \frac{4(n-1)}{n-2} |\nabla \psi|^2 - \frac{4(n-1)}{n-2} n(n+2) W_\epsilon^{\frac{4}{n-2}} \psi^2, \\ J_3 &= \frac{4(n-1)}{n-2} (g_{x_0}^{ik} - \delta_{ik} + h_{ik} - \frac{1}{2} H_{il} H_{kl}) \partial_i W_\epsilon \partial_k W_\epsilon \\ &\quad + \left[ R_{g_{x_0}} - \partial_i \partial_k h_{ik} + \partial_k (H_{ik} \partial_l H_{il}) - \frac{1}{2} \partial_k H_{ik} \partial_l H_{il} + \frac{1}{4} \partial_l H_{ik} \partial_l H_{ik} \right] W_\epsilon^2, \\ J_4 &= \frac{8(n-1)}{n-2} (g_{x_0}^{ik} - \delta_{ik} + H_{ik}) \partial_i W_\epsilon \partial_k \psi + 2(R_{g_{x_0}} - \partial_i \partial_k H_{ik}) W_\epsilon \psi \\ &\quad + R_{g_{x_0}} \psi^2 + \frac{4(n-1)}{n-2} (g_{x_0}^{ik} - \delta_{ik}) \partial_i \psi \partial_k \psi. \end{aligned}$$

We start with  $J_1$ . Rearrange  $J_1$  as

$$\begin{aligned} J_1 &= \frac{8(n-1)}{n-2} \partial_i (\partial_i W_\epsilon \psi) - \frac{8(n-1)}{n-2} \psi \Delta W_\epsilon + \partial_i (W_\epsilon^2 \partial_k h_{ik}) - \partial_k (\partial_i W_\epsilon^2 h_{ik}) \\ &\quad + 2 \left( W_\epsilon \partial_i \partial_k W_\epsilon - \frac{n}{n-2} \partial_i W_\epsilon \partial_k W_\epsilon \right) h_{ik} - \partial_k (W_\epsilon^2 H_{ik} \partial_l H_{il}). \end{aligned}$$

Notice that  $W_\epsilon$  satisfies

$$\psi \Delta W_\epsilon = -\frac{(n-2)^2}{2} \partial_i (W_\epsilon^{\frac{2n}{n-2}} V_i).$$

Thus using  $V_n = 0$  on  $D_\rho$ ,  $H_{in} = h_{in} = 0$ ,  $\text{tr } h = O(|y|^{2d+2})$  in  $B_\rho^+$  and (5.3), we have

$$\begin{aligned} &\int_{B_\rho^+} J_1 dy \\ &= -\frac{8(n-1)}{n-2} \int_{D_\rho} \partial_n W_\epsilon \psi d\sigma + \frac{8(n-1)}{n-2} \int_{\partial^+ B_\rho^+} \psi \partial_i W_\epsilon \frac{y^i}{|y|} d\sigma \\ &\quad + 4(n-1)(n-2) \int_{\partial^+ B_\rho^+} W_\epsilon^{\frac{2n}{n-2}} V_i \frac{y^i}{|y|} d\sigma + \int_{\partial^+ B_\rho^+} (W_\epsilon^2 \partial_k h_{ik} - \partial_k W_\epsilon^2 h_{ik}) \frac{y^i}{|y|} d\sigma \\ &\quad + \int_{B_\rho^+} \frac{2}{n} (W_\epsilon \Delta W_\epsilon - \frac{n}{n-2} |\nabla W_\epsilon|^2) \text{tr } h dy - \int_{\partial^+ B_\rho^+} W_\epsilon^2 H_{ik} \partial_l H_{il} \frac{y^k}{|y|} d\sigma. \end{aligned}$$

Using (5.22) and the expression (2.2) of  $W_\epsilon$ , we estimate

$$\begin{aligned} \int_{\partial^+ B_\rho^+} \partial_i W_\epsilon \psi \frac{y^i}{|y|} d\sigma &\leq C(n, T_c) \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \epsilon^{n-2} \rho^{|\alpha|+2-n}, \\ \int_{\partial^+ B_\rho^+} W_\epsilon^{\frac{2n}{n-2}} V_i \frac{y^i}{|y|} d\sigma &\leq C(n, T_c) \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \epsilon^n \rho^{|\alpha|-n}, \\ \int_{B_\rho^+} (W_\epsilon \Delta W_\epsilon - \frac{n}{n-2} |\nabla W_\epsilon|^2) \text{tr} h \, dy &\leq C(n, T_c) \epsilon^{n-2} \rho^{2d+4-n} \end{aligned}$$

and use  $|\partial H_{ij}| \leq C$  to show

$$\int_{\partial^+ B_\rho^+} W_\epsilon^2 H_{ik} \partial_l H_{il} \frac{y^k}{|y|} d\sigma \leq C(n, T_c) \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \epsilon^{n-2} \rho^{|\alpha|+3-n}.$$

Hence combining the above estimates together, we obtain

$$\begin{aligned} \int_{B_\rho^+} J_1 dy &\leq - \int_{D_\rho} \frac{8(n-1)}{n-2} \partial_n W_\epsilon \psi d\sigma + \int_{\partial^+ B_\rho^+} (W_\epsilon^2 \partial_k h_{ik} - \partial_k W_\epsilon^2 h_{ik}) \frac{y^i}{|y|} d\sigma \\ &\quad + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \epsilon^{n-2} \rho^{|\alpha|+2-n} + C \rho^{2d+4-n} \epsilon^{n-2}. \end{aligned} \quad (5.34)$$

For  $J_2$ , by Proposition 5.4 and (5.24) we have

$$J_2 = -\frac{1}{4} Q_{ik,l} Q_{ik,l} - 2W_\epsilon^{\frac{2n}{n-2}} T_{ik} T_{ik} + \text{div } \xi.$$

By (5.27) a direct computation yields

$$\int_{\partial^+ B_\rho^+} \xi_i \frac{y^i}{|y|} d\sigma \leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 \rho^{2|\alpha|+2-n} \epsilon^{n-2}.$$

From this and Proposition 5.5 we estimate

$$\begin{aligned} &\int_{B_\rho^+} J_2 dy \\ &= -\frac{1}{4} \int_{B_\rho^+} Q_{ik,l} Q_{ik,l} dy - \int_{B_\rho^+} 2W_\epsilon^{\frac{2n}{n-2}} T_{ik} T_{ik} dy + \int_{\partial^+ B_\rho^+} \xi_i \frac{y^i}{|y|} d\sigma - \int_{D_\rho} \xi_n d\sigma \\ &\leq - \int_{D_\rho} \xi_n d\sigma - \frac{n^2}{2(n-1)(n-2)} \int_{D_\rho} \partial_n W_\epsilon W_\epsilon S_{nn}^2 d\sigma \\ &\quad - \frac{1}{4} \lambda^* \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 \epsilon^{n-2} \int_{B_\rho^+} (\epsilon + |y|)^{2|\alpha|+2-2n} dy \end{aligned}$$

$$+ C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 \rho^{2|\alpha|+2-n} \epsilon^{n-2}. \quad (5.35)$$

Observe that when  $|y|$  is sufficiently small, there hold  $|h| \leq C|y|$  and

$$\begin{aligned} |g_{x_0}^{ik} - \delta_{ik}| &\leq C|h|, \\ |g_{x_0}^{ik} - \delta_{ik} + H_{ik}| &\leq C|h|^2 + O(|y|^{d+1}) \leq C|h||y| + O(|y|^{d+1}), \\ |g_{x_0}^{ik} - \delta_{ik} + h_{ik} - \frac{1}{2}H_{il}H_{kl}| &\leq C|h|^3 + O(|y|^{d+2}) \leq C|h|^2|y| + O(|y|^{d+2}). \end{aligned} \quad (5.36)$$

By Proposition 5.6 and Young's inequality, we can bound  $J_3$  and  $J_4$  by

$$\begin{aligned} &J_3 + J_4 \\ &\leq C(n, T_c) \epsilon^{n-2} \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 (\epsilon + |y|)^{2|\alpha|+3-2n} \\ &\quad + C(n, T_c) \epsilon^{n-2} \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| (\epsilon + |y|)^{|\alpha|+d+3-2n} \\ &\quad + C(n, T_c) \epsilon^{n-2} (\epsilon + |y|)^{2d+4-2n} \\ &\leq \frac{1}{2} \lambda^* \epsilon^{n-2} \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 (\epsilon + |y|)^{2|\alpha|+2-2n} + C \epsilon^{n-2} (\epsilon + |y|)^{2d+4-2n}. \end{aligned} \quad (5.37)$$

Consequently, combining the above (5.32), (5.34)-(5.37) and using the decomposition (5.33), we conclude that

$$\begin{aligned} &\int_{B_\rho^+} \left[ \frac{4(n-1)}{n-2} |\nabla(W_\epsilon + \psi)|_{g_{x_0}}^2 + R_{g_{x_0}}(W_\epsilon + \psi)^2 \right] dy + 2(n-1) \int_{D_\rho} h_{g_{x_0}}(W_\epsilon + \psi)^2 d\sigma \\ &\leq \frac{4(n-1)}{n-2} \int_{B_\rho^+} \left[ |\nabla W_\epsilon|^2 + n(n+2) W_\epsilon^{\frac{4}{n-2}} \psi^2 \right] dy \\ &\quad + \int_{\partial^+ B_\rho^+} (W_\epsilon^2 \partial_j h_{ij} - \partial_j W_\epsilon^2 h_{ij}) \frac{y^i}{|y|} d\sigma - \int_{D_\rho} \frac{8(n-1)}{n-2} \partial_n W_\epsilon \psi d\sigma \\ &\quad - \int_{D_\rho} \xi_n d\sigma - \frac{n^2}{2(n-1)(n-2)} \int_{D_\rho} \partial_n W_\epsilon W_\epsilon S_{nn}^2 d\sigma \\ &\quad - \frac{1}{2} \lambda^* \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 \epsilon^{n-2} \int_{B_\rho^+} (\epsilon + |y|)^{2|\alpha|+2-2n} dy \\ &\quad + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \epsilon^{n-2} \rho^{|\alpha|+2-n} + C \epsilon^{n-2} \rho^{2d+4-n}. \end{aligned} \quad (5.38)$$

Testing problem (2.1) with  $W_\epsilon$  and integrating over  $B_\rho^+$ , via integration by parts we obtain

$$\frac{4(n-1)}{n-2} \int_{B_\rho^+} \left[ |\nabla W_\epsilon|^2 + n(n+2) W_\epsilon^{\frac{4}{n-2}} \psi^2 \right] dy$$

$$\begin{aligned}
&= 4n(n-1) \int_{B_\rho^+} W_\epsilon^{\frac{4}{n-2}} \left( W_\epsilon^2 + \frac{n+2}{n-2} \psi^2 \right) dy \\
&\quad + \frac{4(n-1)}{n-2} \int_{\partial^+ B_\rho^+} W_\epsilon \partial_i W_\epsilon \frac{y^i}{|y|} d\sigma - 4(n-1) T_c \int_{D_\rho} W_\epsilon^{\frac{2(n-1)}{n-2}} d\sigma.
\end{aligned}$$

Therefore, plugging this and (5.28) into (5.38) as well as again using (2.1), we obtain the desired assertion.  $\square$

**Proposition 5.8.** *There exists some sufficiently small  $\rho_0$  such that*

$$\begin{aligned}
&4n(n-1) \int_{B_\rho^+} W_\epsilon^{\frac{4}{n-2}} \left( W_\epsilon^2 + \frac{n+2}{n-2} \psi^2 \right) dy \\
&\leq aY_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1}) \left( \int_{B_\rho^+} (W_\epsilon + \psi)^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{n}} + C\epsilon^n \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|-n} \\
&\quad + C\epsilon^n \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 \int_{B_\rho^+} (\epsilon + |y|)^{2|\alpha|+1-2n} dy
\end{aligned}$$

for all  $0 < 2\epsilon \leq \rho \leq \rho_0$ .

*Proof.* Notice that (5.26) gives  $|\psi| \leq C(\epsilon + |y|)W_\epsilon$  in  $B_{2\rho}^+$ . By Lemma 2.1, we get

$$aY_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1}) = 4n(n-1) \left( \int_{\mathbb{R}_+^n} W_\epsilon^{\frac{2n}{n-2}} dx \right)^{\frac{2}{n}}.$$

Together with the fact that  $V_n = 0$  on  $D_{2\rho}$ , the desired estimate can follow the same lines in [11, Propositions 14-15].  $\square$

**Proposition 5.9.** *There exists some sufficiently small  $\rho_0$  such that*

$$\begin{aligned}
&-4(n-1)T_c \int_{D_\rho} W_\epsilon^{\frac{2}{n-2}} \left( W_\epsilon^2 + 2W_\epsilon \psi + \frac{n}{n-2} \psi^2 - \frac{n-2}{8(n-1)^2} W_\epsilon^2 S_{nn}^2 \right) d\sigma \\
&\leq 2(n-1)bY_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1}) \left( \int_{D_\rho} (W_\epsilon + \psi)^{\frac{2(n-1)}{n-2}} d\sigma \right)^{\frac{n-2}{n-1}} \\
&\quad + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|+1-n} \epsilon^{n-1} \\
&\quad + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 \epsilon^{n-1} \rho \int_{D_\rho} (\epsilon + |y|)^{2|\alpha|+2-2n} d\sigma
\end{aligned}$$

for all  $0 < 2\epsilon \leq \rho < \rho_0$ .

*Proof.* Since (5.26) and (5.22) give  $|\psi| \leq C(\epsilon + |y|)W_\epsilon$  and  $|S_{nn}| \leq C(\epsilon + |y|)$  in  $B_{2\rho}^+$ , this assertion can follow the same lines in [13, Proposition 8] (see also [2, (3.23)]) by using

$$-2T_c \left( \int_{\mathbb{R}^{n-1}} W_\epsilon^{\frac{2(n-1)}{n-2}} d\sigma \right)^{\frac{1}{n-1}} = bY_{a,b}(\mathbb{R}_+^n, \mathbb{R}^{n-1})$$

in Lemma 2.1.  $\square$

For simplicity, we denote by  $\Omega_\rho := \Psi_{x_0}(B_\rho^+)$  the coordinate ball of radius  $\rho$  under the Fermi coordinates around  $x_0$ .

**Lemma 5.10.** *If  $0 < \epsilon \ll \rho < \rho_0$  for some sufficiently small  $\rho_0$ , in  $M \setminus \Omega_\rho$  there holds*

$$\begin{aligned} & |\bar{U}_{(x_0, \epsilon)} - \epsilon^{\frac{n-2}{2}} G| \\ & \leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|+2-n} \epsilon^{\frac{n-2}{2}} + C \rho^{d+3-n} |\log \rho| \epsilon^{\frac{n-2}{2}} + C \rho^{1-n} \epsilon^{\frac{n}{2}}. \end{aligned}$$

*Proof.* For  $x \in M \setminus \Omega_\rho$ , let  $y = \Psi_{x_0}^{-1}(x) \in \mathbb{R}_+^n \setminus B_\rho^+$ . In  $M \setminus \Omega_\rho$ , we have

$$\bar{U}_{(x_0, \epsilon)}(x) - \epsilon^{\frac{n-2}{2}} G(x) = \chi_\rho(y) \left[ W_\epsilon(y) + \psi(y) - \epsilon^{\frac{n-2}{2}} G(\Psi_{x_0}(y)) \right]. \quad (5.39)$$

Notice that

$$\begin{aligned} & W_\epsilon(y) - \epsilon^{\frac{n-2}{2}} |y|^{2-n} \\ & = \epsilon^{\frac{n-2}{2}} |y|^{2-n} \left[ \left( 1 + \frac{(1+T_c^2)\epsilon^2}{|y|^2} - \frac{2y^n T_c \epsilon}{|y|^2} \right)^{\frac{2-n}{2}} - 1 \right] \\ & = (n-2)y^n |y|^{-n} T_c \epsilon^{\frac{n}{2}} + O(\epsilon^{\frac{n+2}{2}} |y|^{-n}), \end{aligned}$$

then it yields

$$|W_\epsilon - \epsilon^{\frac{n-2}{2}} |y|^{2-n}| \leq C \epsilon^{\frac{n}{2}} \rho^{1-n} \text{ in } B_{2\rho}^+ \setminus B_\rho^+. \quad (5.40)$$

From this, (5.31) and (5.26), in  $M \setminus \Omega_\rho$  we obtain

$$\begin{aligned} & |\bar{U}_{(x_0, \epsilon)} - \epsilon^{\frac{n-2}{2}} G| \\ & \leq |W_\epsilon - \epsilon^{\frac{n-2}{2}} |y|^{2-n}| + \epsilon^{\frac{n-2}{2}} |G - |y|^{2-n}| + |\psi| \\ & \leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|+2-n} \epsilon^{\frac{n-2}{2}} + \underline{C \rho^{d+3-n} |\log \rho| \epsilon^{\frac{n-2}{2}}} + C \rho^{1-n} \epsilon^{\frac{n}{2}}, \end{aligned}$$

<sup>2</sup>when  $\epsilon \ll \rho < \rho_0$ .  $\square$

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<sup>2</sup>In view of (5.31), the underlined term can be precisely estimated by  $C \rho^{d+3-n} \epsilon^{\frac{n-2}{2}}$  when  $n \geq 5$  and  $C |\log \rho|$  when  $n = 3, 4$ . Since this rough estimate goes through in the later part, we adopt it just for simplicity.

**Lemma 5.11.** *If  $0 < \epsilon \ll \rho < \rho_0$  for some sufficiently small  $\rho_0$ , in  $M \setminus \Omega_\rho$  there holds*

$$\begin{aligned} & \rho^2 \left| \frac{4(n-1)}{n-2} \Delta_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} - R_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} \right| \\ & \leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|+2-n} \epsilon^{\frac{n-2}{2}} + C \rho^{d+3-n} \epsilon^{\frac{n-2}{2}} + C \rho^{1-n} \epsilon^{\frac{n}{2}}. \end{aligned}$$

*Proof.* Since  $\bar{U}_{(x_0, \epsilon)} = \epsilon^{\frac{n-2}{2}} G$  in  $M \setminus \Omega_{2\rho}$ , the estimate is trivial by the definition of  $G$ . Then it suffices to estimate the above inequality in  $\Omega_{2\rho} \setminus \Omega_\rho$ . To see this, by (5.39) we have

$$\begin{aligned} & \Delta_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} - \frac{n-2}{4(n-1)} R_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} \\ & = (\Delta_{g_{x_0}} \chi_\rho)(W_\epsilon + \psi - \epsilon^{\frac{n-2}{2}} |y|^{2-n}) + 2 \langle \nabla \chi_\rho, \nabla (W_\epsilon + \psi - \epsilon^{\frac{n-2}{2}} |y|^{2-n}) \rangle_{g_{x_0}} \\ & \quad - (\Delta_{g_{x_0}} \chi_\rho) \epsilon^{\frac{n-2}{2}} (G - |x|^{2-n}) - 2 \epsilon^{\frac{n-2}{2}} \langle \nabla \chi_\rho, \nabla (G - |x|^{2-n}) \rangle_{g_{x_0}} \\ & \quad + \chi_\rho \left[ \Delta_{g_{x_0}} (W_\epsilon + \psi) - \frac{n-2}{4(n-1)} R_{g_{x_0}} (W_\epsilon + \psi) \right] \\ & = I_1 + I_2 + I_3, \end{aligned}$$

where  $I_i (i = 1, 2, 3)$  denotes the quantity in each corresponding line. By using (5.40) and  $|\rho^2 \Delta_{g_{x_0}} \chi_\rho| + |\rho \nabla \chi_\rho|_{g_{x_0}} \leq C$ , we get

$$\rho^2 I_1 \leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|+2-n} \epsilon^{\frac{n-2}{2}} + C \rho^{1-n} \epsilon^{\frac{n}{2}}.$$

Similarly (5.31) implies

$$\rho^2 I_2 \leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|+2-n} \epsilon^{\frac{n-2}{2}} + C \rho^{d+3-n} \epsilon^{\frac{n-2}{2}} + C \rho^{1-n} \epsilon^{\frac{n}{2}}.$$

For  $I_3$ , applying the property (5.26) of  $\psi$  and Proposition 5.6, we get

$$|R_{g_{x_0}} (W_\epsilon + \psi)| \leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|-n} \epsilon^{\frac{n-2}{2}} + C \rho^{d+1-n} \epsilon^{\frac{n-2}{2}}$$

and

$$\begin{aligned} & |\Delta_{g_{x_0}} (W_\epsilon + \psi)| \\ & \leq |(\Delta_{g_{x_0}} - \Delta_{\mathbb{R}^n})(W_\epsilon + \psi)| + C \epsilon^{\frac{n+2}{2}} \rho^{-n-2} + C \epsilon^{\frac{n-2}{2}} \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|-n} \\ & \leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|-n} \epsilon^{\frac{n-2}{2}} + C \rho^{d+1-n} \epsilon^{\frac{n-2}{2}} + C \rho^{-n-2} \epsilon^{\frac{n+2}{2}} \end{aligned}$$

Therefore

$$\rho^2 I_3 \leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|+2-n} \epsilon^{\frac{n-2}{2}} + C \rho^{d+3-n} \epsilon^{\frac{n-2}{2}} + C \rho^{-n} \epsilon^{\frac{n+2}{2}}.$$

Collecting all the above estimates on  $I_1$ - $I_3$ , we get the desired assertion.  $\square$

We now arrive at the key Proposition 5.12.

**Proposition 5.12.** *If  $0 < \epsilon \ll \rho < \rho_0$  for some sufficiently small  $\rho_0$ , there holds*

$$\begin{aligned} & \int_M \left[ \frac{4(n-1)}{n-2} |\nabla \bar{U}_{(x_0, \epsilon)}|_{g_{x_0}}^2 + R_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 \right] d\mu_{g_{x_0}} + 2(n-1) \int_{\partial M} h_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 d\sigma_{g_{x_0}} \\ & \leq Y_{a,b}(\mathbb{R}^n, \mathbb{R}^{n-1}) \left[ a \left( \int_M \bar{U}_{(x_0, \epsilon)}^{\frac{2n}{n-2}} d\mu_{g_{x_0}} \right)^{\frac{n-2}{n}} \right. \\ & \quad \left. + 2(n-1)b \left( \int_{\partial M} \bar{U}_{(x_0, \epsilon)}^{\frac{2(n-1)}{n-2}} d\sigma_{g_{x_0}} \right)^{\frac{n-2}{n-1}} \right] \\ & \quad - \epsilon^{n-2} \mathcal{I}(x_0, \rho) - \frac{1}{C} \eta_{\mathcal{Z}^c}(x_0) \lambda^* \epsilon^{n-2} \int_{B_\rho^+} |W_{g_0}(y)|_{g_0}^2 (\epsilon + |y|)^{6-2n} dy \\ & \quad - \frac{1}{C} \eta_{\mathcal{Z}^c}(x_0) \lambda^* \epsilon^{n-2} \int_{D_\rho} |\hat{\pi}_{g_0}(y)|_{g_0}^2 (\epsilon + |y|)^{5-2n} d\sigma + C \rho^{2d+4-n} |\log \rho|^2 \epsilon^{n-2} \\ & \quad + C \left( \frac{\epsilon}{\rho} \right)^{n-2} \frac{1}{\log(\rho/\epsilon)} + C^* \left( \frac{\epsilon}{\rho} \right)^{n-1}, \end{aligned}$$

where  $\eta_{\mathcal{Z}^c}$  is the characteristic function of  $\mathcal{Z}^c = \partial M \setminus \mathcal{Z}$  defined on  $\partial M$  and  $C, C^*$  depend on  $n, g_0, T_c, \rho_0$ .

*Proof.* Observe that

$$\begin{aligned} & \int_{M \setminus \Omega_\rho} \left[ \frac{4(n-1)}{n-2} |\nabla \bar{U}_{(x_0, \epsilon)}|_{g_{x_0}}^2 + R_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 \right] d\mu_{g_{x_0}} + 2(n-1) \int_{\partial M \setminus \Omega_\rho} h_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 d\sigma_{g_{x_0}} \\ & = \int_{M \setminus \Omega_\rho} \left( -\frac{4(n-1)}{n-2} \Delta_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} + R_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 \right) (\bar{U}_{(x_0, \epsilon)} - \epsilon^{\frac{n-2}{2}} G) d\mu_{g_{x_0}} \\ & \quad + \frac{4(n-1)}{n-2} \int_{\partial(M \setminus \Omega_\rho)} \left[ \frac{\partial \bar{U}_{(x_0, \epsilon)}}{\partial \nu_{g_{x_0}}} \bar{U}_{(x_0, \epsilon)} + \epsilon^{\frac{n-2}{2}} \left( \bar{U}_{(x_0, \epsilon)} \frac{\partial G}{\partial \nu_{g_{x_0}}} - G \frac{\partial \bar{U}_{(x_0, \epsilon)}}{\partial \nu_{g_{x_0}}} \right) \right] d\sigma_{g_{x_0}} \\ & \quad + 2(n-1) \int_{\partial M \setminus \Omega_\rho} h_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 d\sigma_{g_{x_0}} \\ & = II_1 + II_2 + II_3, \end{aligned}$$

where  $II_i (i = 1, 2, 3)$  denotes the quantity in each corresponding line on the right hand side of the first identity. By Lemmas 5.10 and 5.11, we get

$$\sup_{M \setminus \Omega_\rho} \left[ |\bar{U}_{(x_0, \epsilon)} - \epsilon^{\frac{n-2}{2}} G| + \rho^2 \left| \frac{4(n-1)}{n-2} \Delta_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} - R_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} \right| \right]$$

$$\leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|+2-n} \epsilon^{\frac{n-2}{2}} + C \rho^{d+3-n} |\log \rho| \epsilon^{\frac{n-2}{2}} + C \rho^{1-n} \epsilon^{\frac{n}{2}}.$$

From this, one can estimate  $II_1$  as

$$\begin{aligned} II_1 &= \int_{\Omega_{2\rho} \setminus \bar{\Omega}_\rho} \left( -\frac{4(n-1)}{n-2} \Delta_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} + R_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} \right) (\bar{U}_{(x_0, \epsilon)} - \epsilon^{\frac{n-2}{2}} G) d\mu_{g_{x_0}} \\ &\leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 \rho^{2|\alpha|+2-n} \epsilon^{n-2} + C \rho^{2d+4-n} |\log \rho|^2 \epsilon^{n-2} + C \rho^{-n} \epsilon^n. \end{aligned} \quad (5.41)$$

For  $II_2$ , we divide the integral into two parts  $II_2 = II_2^{(1)} + II_2^{(2)}$  according to  $\partial(M \setminus \Omega_\rho) = (\partial M \setminus \Omega_\rho) \cup (\partial \Omega_\rho \setminus \partial M)$ . Namely  $II_2^{(1)}$  is the integral over  $\partial M \setminus \Omega_\rho$  while  $II_2^{(2)}$  is over  $\partial \Omega_\rho \setminus \partial M$ . Let us deal with  $II_2^{(1)}$  first. In  $\partial M \setminus \Omega_\rho$ , by Lemma 5.2, (2.1), (5.30) and (5.2), we have

$$\begin{aligned} &\sup_{\partial M \cap (\Omega_{2\rho} \setminus \bar{\Omega}_\rho)} \left| \frac{\partial \bar{U}_{(x_0, \epsilon)}}{\partial \nu_{g_{x_0}}} \right| \\ &\leq \sup_{\partial M \cap (\Omega_{2\rho} \setminus \bar{\Omega}_\rho)} \left[ |\partial_n W_\epsilon + \partial_n \psi| + \epsilon^{\frac{n-2}{2}} \left| \frac{\partial G}{\partial \nu_{g_{x_0}}} \right| \right] \\ &\leq C \epsilon^{\frac{n}{2}} \rho^{-n} + C \epsilon^{\frac{n}{2}} \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|-n} + C \epsilon^{\frac{n-2}{2}} \rho^{2d+3-n}. \end{aligned}$$

Using (5.30), Lemma 5.10, (5.2) and  $\bar{U}_{(x_0, \epsilon)} = \epsilon^{\frac{n-2}{2}} G$  in  $M \setminus \Omega_{2\rho}$ , we have

$$\begin{aligned} &II_2^{(1)} + II_3 \\ &= \frac{4(n-1)}{n-2} \int_{\partial M \setminus \bar{\Omega}_\rho} \left[ \frac{\partial \bar{U}_{(x_0, \epsilon)}}{\partial \nu_{g_{x_0}}} \bar{U}_{(x_0, \epsilon)} + \epsilon^{\frac{n-2}{2}} \left( \bar{U}_{(x_0, \epsilon)} \frac{\partial G}{\partial \nu_{g_{x_0}}} - G \frac{\partial \bar{U}_{(x_0, \epsilon)}}{\partial \nu_{g_{x_0}}} \right) \right] d\sigma_{g_{x_0}} \\ &\quad + II_3 \\ &= \frac{4(n-1)}{n-2} \int_{\partial M \cap (\Omega_{2\rho} \setminus \bar{\Omega}_\rho)} \frac{\partial \bar{U}_{(x_0, \epsilon)}}{\partial \nu_{g_{x_0}}} (\bar{U}_{(x_0, \epsilon)} - \epsilon^{\frac{n-2}{2}} G) d\sigma_{g_{x_0}} \\ &\quad + 2(n-1) \int_{\partial M \cap (\Omega_{2\rho} \setminus \bar{\Omega}_\rho)} h_{g_{x_0}} \bar{U}_{(x_0, \epsilon)} (\bar{U}_{(x_0, \epsilon)} - \epsilon^{\frac{n-2}{2}} G) d\sigma_{g_{x_0}} \\ &\leq C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 \rho^{2|\alpha|+1-n} \epsilon^{n-1} + |\partial^\alpha h_{ab}| \rho^{|\alpha|+1-n} \epsilon^{n-1} \\ &\quad + C \rho^{d+2-n} \epsilon^{n-1} + C \rho^{-n} \epsilon^n + C \rho^{2d+4-n} |\log \rho| \epsilon^{n-2}. \end{aligned} \quad (5.42)$$

Next we start to estimate  $II_2^{(2)}$  whose integral domain is  $\partial \Omega_\rho \setminus \partial M$ . It is not hard to verify that the outward unit normal  $\nu_{g_{x_0}}$  on  $\partial \Omega_\rho \setminus \partial M := \Psi_{x_0}(\partial^+ B_\rho^+)$  is given by

$$\nu_{g_{x_0}} = \frac{g_{x_0}^{ik} y^k}{\|y\|} \partial_{y^i} \text{ for } y \in \partial^+ B_\rho^+,$$



where  $\|y\|^2 := g_{x_0}^{kl} y^k y^l = \rho^2(1 + C|h|)$  on  $\partial^+ B_\rho^+$ . Note that  $\bar{U}_{(x_0, \epsilon)} = W_\epsilon + \psi$  on  $\partial^+ B_\rho^+$ , by (5.36) we estimate

$$\begin{aligned}
& \int_{\partial\Omega_\rho \setminus \partial M} \frac{\partial \bar{U}_{(x_0, \epsilon)}}{\partial \nu_{g_{x_0}}} \bar{U}_{(x_0, \epsilon)} d\sigma_{g_{x_0}} \\
&= - \int_{\partial^+ B_\rho^+} g_{x_0}^{ij} \partial_i \bar{U}_{(x_0, \epsilon)} \frac{y^j}{\|y\|} \bar{U}_{(x_0, \epsilon)} d\sigma + O(\rho^{2d+4-n} \epsilon^{n-2}) \\
&= \int_{\partial^+ B_\rho^+} (-\partial_i \bar{U}_{(x_0, \epsilon)} + \partial_j \bar{U}_{(x_0, \epsilon)} h_{ij}) \frac{y^i}{|y|} (1 + C|h|) \bar{U}_{(x_0, \epsilon)} d\sigma \\
&\quad + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 \rho^{2|\alpha|+2-n} \epsilon^{n-2} + O(\rho^{2d+4-n} \epsilon^{n-2}) \\
&\leq \int_{\partial^+ B_\rho^+} (-\partial_i W_\epsilon + \partial_j W_\epsilon h_{ij}) \frac{y^i}{|y|} W_\epsilon d\sigma + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|+2-n} \epsilon^{n-2} \\
&\quad + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 \rho^{2|\alpha|+2-n} \epsilon^{n-2} + C \rho^{2d+4-n} \epsilon^{n-2}. \tag{5.43}
\end{aligned}$$

Similarly we have

$$\begin{aligned}
& \epsilon^{\frac{n-2}{2}} \int_{\partial\Omega_\rho \setminus \partial M} \left( \bar{U}_{(x_0, \epsilon)} \frac{\partial G}{\partial \nu_{g_{x_0}}} - G \frac{\partial \bar{U}_{(x_0, \epsilon)}}{\partial \nu_{g_{x_0}}} \right) d\sigma_{g_{x_0}} \\
&\leq - \epsilon^{\frac{n-2}{2}} \int_{\partial^+ B_\rho^+} (\bar{U}_{(x_0, \epsilon)} \partial_i G - G \partial_i \bar{U}_{(x_0, \epsilon)}) \frac{y^i}{|y|} (1 + C|h|) d\sigma \\
&\quad + \epsilon^{\frac{n-2}{2}} \int_{\partial^+ B_\rho^+} h_{ij} \frac{y^i}{|y|} (1 + C|h|) (\bar{U}_{(x_0, \epsilon)} \partial_j G - G \partial_j \bar{U}_{(x_0, \epsilon)}) d\sigma \\
&\quad + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 \rho^{2|\alpha|+2-n} \epsilon^{n-2} + C \rho^{2d+4-n} \epsilon^{n-2} \\
&\leq - \epsilon^{\frac{n-2}{2}} \int_{\partial^+ B_\rho^+} (W_\epsilon \partial_i G - G \partial_i W_\epsilon) \frac{y^i}{|y|} d\sigma \\
&\quad + \epsilon^{\frac{n-2}{2}} \int_{\partial^+ B_\rho^+} h_{ij} \frac{y^i}{|y|} (W_\epsilon \partial_j G - G \partial_j W_\epsilon) d\sigma \\
&\quad + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|+2-n} \epsilon^{n-2} + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 \rho^{2|\alpha|+2-n} \epsilon^{n-2} \\
&\quad + C \rho^{2d+4-n} \epsilon^{n-2}. \tag{5.44}
\end{aligned}$$

From (5.31) and (5.40), on  $\partial^+ B_\rho^+$  we get

$$\epsilon^{\frac{n-2}{2}} |\partial_i W_\epsilon G - \partial_i G W_\epsilon|$$

$$\begin{aligned}
& \leq |\partial_i W_\epsilon(\epsilon^{\frac{n-2}{2}} G - W_\epsilon)| + |W_\epsilon \partial_i(\epsilon^{\frac{n-2}{2}} G - W_\epsilon)| \\
& \leq C \epsilon^{n-2} \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|+3-2n} + C \epsilon^{n-2} \rho^{d+4-2n} |\log \rho| + C \epsilon^{n-1} \rho^{2-2n}
\end{aligned}$$

and then

$$\begin{aligned}
& \epsilon^{\frac{n-2}{2}} \int_{\partial^+ B_\rho^+} h_{ij} \frac{y^i}{|y|} (W_\epsilon \partial_j G - G \partial_j W_\epsilon) d\sigma \\
& \leq C \epsilon^{n-2} \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 \rho^{2|\alpha|+2-n} + C \rho^{2d+4-n} |\log \rho| \epsilon^{n-2} + C \epsilon^{n-1} \rho^{2-n}. \quad (5.45)
\end{aligned}$$

Hence plugging (5.45) into (5.44), we obtain

$$\begin{aligned}
& \epsilon^{\frac{n-2}{2}} \int_{\partial \Omega_\rho \setminus \partial M} (\bar{U}_{(x_0, \epsilon)} \frac{\partial G}{\partial \nu_{g_{x_0}}} - G \frac{\partial G}{\partial \nu_{g_{x_0}}}) d\sigma_{g_{x_0}} \\
& \leq -\epsilon^{\frac{n-2}{2}} \int_{\partial^+ B_\rho^+} (W_\epsilon \partial_i G - G \partial_i W_\epsilon) \frac{y^i}{|y|} d\sigma + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|+2-n} \epsilon^{n-2} \\
& \quad + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 \rho^{2|\alpha|+2-n} \epsilon^{n-2} + C \rho^{2d+4-n} |\log \rho| \epsilon^{n-2} + C \epsilon^{n-1} \rho^{2-n}. \quad (5.46)
\end{aligned}$$

Consequently combining (5.43) and (5.46), we can get

$$\begin{aligned}
II_2^{(2)} & \leq -\frac{4(n-1)}{n-2} \int_{\partial^+ B_\rho^+} \left[ \partial_i W_\epsilon W_\epsilon - \partial_j W_\epsilon W_\epsilon h_{ij} + \epsilon^{\frac{n-2}{2}} (W_\epsilon \partial_i G - G \partial_i W_\epsilon) \right] \frac{y^i}{|y|} d\sigma \\
& \quad + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|+2-n} \epsilon^{n-2} + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 \rho^{2|\alpha|+2-n} \epsilon^{n-2} \\
& \quad + C \rho^{2d+4-n} |\log \rho| \epsilon^{n-2} + C \epsilon^{n-1} \rho^{2-n}. \quad (5.47)
\end{aligned}$$

Therefore collecting the estimates (5.41) for  $II_1$ , (5.42) for  $II_2^{(1)} + II_3$  and (5.47) for  $II_2^{(2)}$  together, when  $\epsilon \ll \rho < \rho_0$  we obtain

$$\begin{aligned}
& \int_{M \setminus \Omega_\rho} \left[ \frac{4(n-1)}{n-2} |\nabla \bar{U}_{(x_0, \epsilon)}|_{g_{x_0}}^2 + R_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 \right] d\mu_{g_{x_0}} + 2(n-1) \int_{\partial M \setminus \Omega_\rho} h_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 d\sigma_{g_{x_0}} \\
& \leq \frac{4(n-1)}{n-2} \int_{\partial^+ B_\rho^+} \left[ -\partial_i W_\epsilon W_\epsilon + \partial_j W_\epsilon W_\epsilon h_{ij} - \epsilon^{\frac{n-2}{2}} (W_\epsilon \partial_i G - G \partial_i W_\epsilon) \right] \frac{y^i}{|y|} d\sigma \\
& \quad + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|+2-n} \epsilon^{n-2} + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 \rho^{2|\alpha|+2-n} \epsilon^{n-2} \\
& \quad + C \rho^{2d+4-n} |\log \rho|^2 \epsilon^{n-2} + C \rho^{2-n} \epsilon^{n-1}. \quad (5.48)
\end{aligned}$$

Finally since  $d\mu_{g_{x_0}} = (1 + O(|y|^{2d+2}))dy$  and  $d\sigma_{g_{x_0}} = (1 + O(|y|^{2d+2}))d\sigma$  under the Fermi coordinates around  $x_0 \in \partial M$ , noticing that Propositions 5.7-5.9 and (5.48) give the estimates of energy  $E[\bar{U}_{(x_0, \epsilon)}]$  in the interior of  $B_\rho^+ = \Psi_{x_0}^{-1}(\Omega_\rho)$  and in the exterior of  $\Omega_\rho$  respectively, we conclude that

$$\begin{aligned}
& \int_M \left[ \frac{4(n-1)}{n-2} |\nabla \bar{U}_{(x_0, \epsilon)}|_{g_{x_0}}^2 + R_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 \right] d\mu_{g_{x_0}} + 2(n-1) \int_{\partial M} h_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 d\sigma_{g_{x_0}} \\
& \leq Y_{a,b}(\mathbb{R}^n, \mathbb{R}^{n-1}) \left[ a \left( \int_M \bar{U}_{(x_0, \epsilon)}^{\frac{2n}{n-2}} d\mu_{g_{x_0}} \right)^{\frac{n-2}{n}} \right. \\
& \quad \left. + 2(n-1)b \left( \int_{\partial M} \bar{U}_{(x_0, \epsilon)}^{\frac{2(n-1)}{n-2}} d\sigma_{g_{x_0}} \right)^{\frac{n-2}{n-1}} \right] \\
& + \int_{\partial^+ B_\rho^+} (W_\epsilon^2 \partial_j h_{ij} + \frac{n}{n-2} \partial_j W_\epsilon^2 h_{ij}) \frac{y^i}{|y|} d\sigma \\
& - \frac{4(n-1)}{n-2} \epsilon^{\frac{n-2}{2}} \int_{\partial^+ B_\rho^+} (W_\epsilon \partial_i G - G \partial_i W_\epsilon) \frac{y^i}{|y|} d\sigma \\
& - \frac{1}{4} \lambda^* \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 \epsilon^{n-2} \int_{B_\rho^+} (\epsilon + |y|)^{2|\alpha|+2-2n} dy \\
& + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}|^2 \epsilon^{n-1} \rho \int_{D_\rho} (\epsilon + |y|)^{2|\alpha|+2-2n} d\sigma \\
& + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|+2-n} \epsilon^{n-2} + C \rho^{2d+4-n} |\log \rho|^2 \epsilon^{n-2} \\
& + C \epsilon^{n-1} \rho^{2-n}, \tag{5.49}
\end{aligned}$$

where we have used the following estimate:

$$\begin{aligned}
& \epsilon^n \int_{B_\rho^+} (\epsilon + |y|)^{2|\alpha|+1-2n} dy \\
& \leq C \epsilon^{n-1} \int_{B_\rho^+} (\epsilon + |y|)^{2|\alpha|+2-2n} dy \leq \frac{\lambda^*}{4} \epsilon^{n-2} \int_{B_\rho^+} (\epsilon + |y|)^{2|\alpha|+2-2n} dy
\end{aligned}$$

by choosing  $\epsilon \ll \rho < \rho_0$ . By (5.31) and the expression (2.2) of  $W_\epsilon$ , we get

$$\begin{aligned}
& \int_{\partial^+ B_\rho^+} (W_\epsilon^2 \partial_j h_{ij} + \frac{n}{n-2} \partial_j W_\epsilon^2 h_{ij}) \frac{y^i}{|y|} d\sigma \\
& - \frac{4(n-1)}{n-2} \epsilon^{\frac{n-2}{2}} \int_{\partial^+ B_\rho^+} (W_\epsilon \partial_i G - G \partial_i W_\epsilon) \frac{y^i}{|y|} d\sigma \\
& \leq -\epsilon^{n-2} \mathcal{I}(x_0, \rho) + C \sum_{a,b=1}^{n-1} \sum_{|\alpha|=1}^d |\partial^\alpha h_{ab}| \rho^{|\alpha|+1-n} \epsilon^{n-1} + C \epsilon^{n-1} \rho^{1-n}. \tag{5.50}
\end{aligned}$$

Notice that

$$|W_{g_0}(x)|_{g_0} = f_{x_0}^{\frac{4}{n-2}} |W_{g_{x_0}}(x)|_{g_{x_0}} \leq C|\partial^2 h| + |\partial h| \quad \text{in } M$$

and

$$|\mathring{\pi}_{g_0}(x)|_{g_0} = f_{x_0}^{\frac{2}{n-2}} |W_{g_{x_0}}(x)|_{g_{x_0}} \leq C|\partial h| \quad \text{on } \partial M.$$

By choosing  $\rho_0$  small enough with all  $\rho < \rho_0$ , it is not hard to show that

$$C\epsilon^{n-1}\rho \int_{D_\rho} (\epsilon + |y|)^{2|\alpha|+2-2n} d\sigma \leq \frac{\lambda^*}{8}\epsilon^{n-2} \int_{B_\rho^+} (\epsilon + |y|)^{2|\alpha|+2-2n} dy.$$

Recall that we define by  $\mathcal{Z}$  the set of all points  $x_0 \in \partial M$  satisfying

$$\limsup_{x \rightarrow x_0} d_{g_0}(x, x_0)^{2-d} |W_{g_0}(x)|_{g_0} = \limsup_{x \rightarrow x_0} d_{g_0}(x, x_0)^{1-d} |\mathring{\pi}_{g_0}(x)|_{g_0} = 0.$$

From these estimates, (5.49) and (5.50), a similar arguments in [2, Corollary 3.10] yields

$$\begin{aligned} & \int_M \left[ \frac{4(n-1)}{n-2} |\nabla \bar{U}_{(x_0, \epsilon)}|_{g_{x_0}}^2 + R_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 \right] d\mu_{g_{x_0}} + 2(n-1) \int_{\partial M} h_{g_{x_0}} \bar{U}_{(x_0, \epsilon)}^2 d\sigma_{g_{x_0}} \\ & \leq Y_{a,b}(\mathbb{R}^n, \mathbb{R}^{n-1}) \left[ a \left( \int_M \bar{U}_{(x_0, \epsilon)}^{\frac{2n}{n-2}} d\mu_{g_{x_0}} \right)^{\frac{n-2}{n}} \right. \\ & \quad \left. + 2(n-1)b \left( \int_{\partial M} \bar{U}_{(x_0, \epsilon)}^{\frac{2(n-1)}{n-2}} d\sigma_{g_{x_0}} \right)^{\frac{n-2}{n-1}} \right] \\ & \quad - \epsilon^{n-2} \mathcal{I}(x_0, \rho) - \frac{1}{C} \eta_{\mathcal{Z}^c}(x_0) \lambda^* \epsilon^{n-2} \int_{B_\rho^+} |W_{g_0}(y)|_{g_0}^2 (\epsilon + |y|)^{6-2n} dy \\ & \quad - \frac{1}{C} \eta_{\mathcal{Z}^c}(x_0) \lambda^* \epsilon^{n-2} \int_{D_\rho} |\mathring{\pi}_{g_0}(y)|_{g_0}^2 (\epsilon + |y|)^{5-2n} d\sigma + C^* \rho^{2d+4-n} |\log \rho|^2 \epsilon^{n-2} \\ & \quad + C \left( \frac{\epsilon}{\rho} \right)^{n-2} \frac{1}{\log(\rho/\epsilon)} + C \left( \frac{\epsilon}{\rho} \right)^{n-1}, \end{aligned}$$

by recalling that  $\eta_{\mathcal{Z}^c}$  is the characteristic function of  $\mathcal{Z}^c = \partial M \setminus \mathcal{Z}$ .  $\square$

Next we describe the continuity of  $\mathcal{I}(x_0, \rho)$  over  $\mathcal{Z}$  as in [2, Proposition 3.11] and some characterization of its limit as  $\rho \rightarrow 0$  (cf. [12, Proposition 4.3]). We restate them here for convenience.

**Proposition 5.13.** *The functions  $\mathcal{I}(x_0, \rho)$  converge to a continuous function  $\mathcal{I}(x_0) : \mathcal{Z} \rightarrow \mathbb{R}$  uniformly for all  $x_0 \in \mathcal{Z}$ , as  $\rho \rightarrow 0$ .*

**Proposition 5.14.** *Let  $x_0 \in \mathcal{Z}$  and consider inverted coordinates  $\Phi : y \in \bar{M} \setminus \{x_0\} \mapsto z := y/|y|^2$ , where  $y = (y^1, \dots, y^n)$  are Fermi coordinates centered at  $x_0$ . If we define the metric  $\bar{g}_{x_0} = \Phi_*(G_{x_0}^{4/(n-2)} g_{x_0})$  on  $\bar{M} \setminus \{x_0\}$ , then the following statements hold:*

(i)  *$(\bar{M} \setminus \{x_0\}, \bar{g})$  is an asymptotically flat manifold with order  $d+1 > \frac{n-2}{2}$  (in the sense of Definition 1.2), and satisfies  $R_{\bar{g}} \equiv 0$  and  $h_{\bar{g}} \equiv 0$ .*

(ii) We have

$$\mathcal{I}(x_0) = \lim_{R \rightarrow \infty} \left[ \int_{\partial^+ B_R^+} \frac{z^i}{|z|} \partial_{z^j} \bar{g}_{x_0}(\partial_{z^i}, \partial_{z^j}) d\sigma - \int_{\partial^+ B_R^+} \frac{z^i}{|z|} \partial_{z^i} \bar{g}_{x_0}(\partial_{z^j}, \partial_{z^j}) d\sigma \right].$$

In particular,  $\mathcal{I}(x_0)$  is the mass  $m(\bar{g}_{x_0})$  of  $(\bar{M} \setminus \{x_0\}, \bar{g}_{x_0})$ .

*Proof of Theorem 1.3.* (i) When  $\partial M \setminus \mathcal{Z} \neq \emptyset$ , we choose  $x_0 \in \partial M \setminus \mathcal{Z}$ . Then the desired assertion follows from Proposition 5.13.

(ii) Assume that  $\mathcal{I}(x_0) > 0$  for some  $x_0 \in \mathcal{Z}$ , it follows from Proposition 5.13 that

$$\mathcal{I}(x_0, \rho) > C^* \rho^{2d+4-n} |\log \rho|^2$$

for all  $0 < \rho < \rho_0$ , where  $\rho_0, C^*$  are the positive constants in Proposition 5.13. Based on the key estimate in Proposition 5.13, Theorem 1.3 follows the same lines of [2, Proposition 3.7].  $\square$

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